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One-loop Yukawas on intersecting branes

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Abstract: We calculate Yukawa interactions at one-loop on intersecting D6 branes. We demonstrate the non-renormalization theorem in supersymmetric configurations, and show how Yukawa β-functions may be extracted. In addition to the usual logarithmic running, we find the power-law dependence on the infra-red cut-off associated with Kaluza-Klein modes. Our results may also be used to evaluate coupling renormalization in non-supersymmetric cases.

Keywords: D-branes, Compactification and String Models.

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1. Introduction

Open string models based on intersecting D-branes (see [1 – 4] for recent reviews) have stimulated new approaches in a number of areas of unification physics. They have also proven to be a useful laboratory for testing ideas initially presented in extra-dimensional field theories, without any of the concomitant renormalisability or finiteness problems.

Chief amongst these, for the purposes of this paper, are ideas concerning Yukawa couplings and their possible hierarchies. One striking feature of intersecting branes is their natural replication of families at different intersections, which naturally leads to the idea that Yukawa hierarchies have a geometrical origin [5, 6]: small Yukawa couplings can arise if different families are located at different intersections, and the couplings can be exponentially suppressed by world-sheet instantons whose actions are the areas of the 'Yukawa triangles'. This type of picture has its equivalent in closed string orbifold models where the couplings between twisted states at different fixed points are similarly suppressed [7 – 10]. There has been significant interest in the phenomenological implications of such a set-up and many extra-dimensional ideas (for example, the contribution of Kaluza-Klein states to flavour changing [11]) found their natural realisation here. On a more formal level, Yukawa couplings have important applications in understanding brane recombination processes — for instance, in the small instanton phase transition of a 3-brane dissolving on a 7-brane, which would be a D6-brane recombination process in the T-dual picture [12].

Going beyond tree-level, there are ideas about Yukawa couplings that have not yet been addressed in a string theory context. One that will concern us here is power law running. Any model with extra dimensions enjoys the possibility of greatly accelerated renormalisation group running of gauge and Yukawa couplings due to the contribution of Kaluza-Klein states. This effect was originally suggested in the context of large extra dimensions [13 – 15], based on the ultra-violet cut-off dependence of couplings at one loop in field theory with extra dimensions. It has since been explored for gauge couplings in extra-dimensional field theory (despite its non-renormalisability) by calculating one-loop vacuum polarization diagrams off-shell and computing their energy-scale dependence in various regularisation schemes [16, 17]. A genuine energy-scale dependence has been established and agrees (up to subdominant scheme dependent pieces) with the IR-cut-off dependence in various string compactifications [18, 19].

There are a number of reasons why a similarly direct comparison of power law running has not yet been done for the Yukawa couplings, most notably the fact that the tree-level Yukawa couplings are set (at least in the intersecting brane picture) by non-perturbative classical world sheet instantons. There is no easy prescription for inserting these non-perturbative tree-level couplings into one-loop diagrams other than simply truncating to the extra dimensional field theory, which would be begging the question. If we want to derive power law running of Yukawas from intersecting brane configurations, we have to make sure that the *one-loop* classical instantons give the expected factors of *tree-level* Yukawa couplings in the field theory limits of the string diagram. An additional complicating factor is the technical difficulty of calculating the one-loop correlation functions that are required to describe interactions between fields living at intersections.

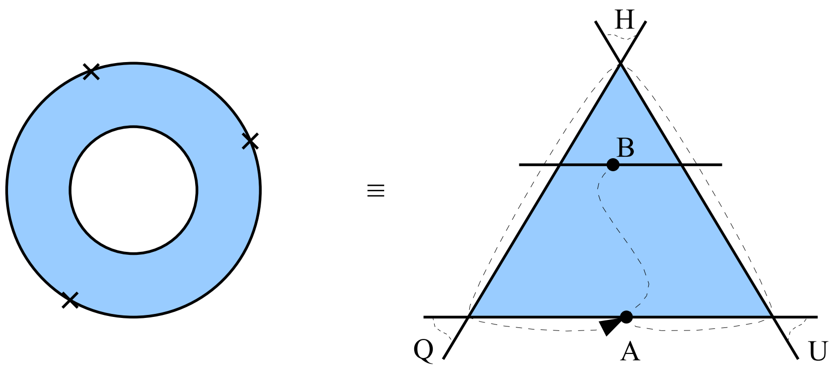

Figure 1: The annulus diagram corresponds to taking an open string stretched between two branes as shown, and moving one end around the Yukawa triangle.

So, with this aim in mind, we present the calculation of Yukawa couplings at one-loop on intersecting branes. (A more general motivation for the study in the present paper is simply that we would like to know how to do perturbation theory at one-loop on intersecting branes, anyway.) Our main focus will be the annulus diagram with no orbifold twists, shown in figures 1 and 2.

We will for concreteness consider D6 branes (although the techniques can easily be extended to other configurations) intersecting at angles in a factorisable torus $T_2 \times T_2 \times T_2$ whose sub-tori may be tilted and may contain orientifold planes. A general $\mathcal{N} = 1$ set-up will usually involve orbifolds and orientifolds so that there will also be twisted diagrams if the D-branes go through orbifold fixed points, and there will be Möbius strip diagrams as well. All of the techniques we are going to describe can be used for those diagrams as well as the untwisted annulus. However, one can get all the extra information required about these additional diagrams by factorising on the one-loop partition function, and so it is possible to present results quite generally in terms of the latter. The nett effect of the other diagrams is, as one would expect, simply to add new twisted sectors or to project out states in the spectrum. Consequently, we will focus on the untwisted diagram, but our results will are easily converted to the twisted case.

The first figure spells out the physical principle of the calculation, discussed in ref. [20]. This is to take a string stretched between two branes as shown and keep one end (B) fixed on a particular brane, whilst the opposing end (A) sweeps out a Yukawa triangle. Quark and Higgs states are deposited at each vertex of the triangle as the endpoint A switches from one brane to the next. The corresponding worldsheet diagram is then the annulus with two fermion and two boson vertex operator insertions on the boundary. There is no constraint on the relative positioning of the B brane (although the usual rule that the action goes as the square of the brane separation will continue to be obeyed), and it may be one of the other three branes (in fact, all renormalisation diagrams in the effective field theory will be of this type, and diagrams involving four branes generically separated correspond to string scale masses circulating in the loop).

In the second figure we have mapped the annulus world-sheet to the rectangular domain shown, which has width $\frac{1}{2}$ and height it. The 'branch-cuts' are there to indicate that many correlation functions (for example those involving ∂X) will get a phase as they go round the vertex. This takes account of the change in angle of the allowed motion of the string end-point A as it switches from one brane to another at the intersection. The regions between vertices then correspond to separate branes (with corresponding phases of ∂X), so the region with no branch cut on the left is the *same brane as* (or at least parallel to) the brane on the right. The technology of mapping a triangle with a hole in it to a fundamental domain such as this is known as Schwarz-Christoffel mapping. It was discussed in ref. [21] for the tree level case, so we will not dwell on it here except to say that we expect the correlation functions to involve products of elliptic functions in the one loop case. Note that it is possible to have branes on the right that are not parallel to any on the left in which case one has branch-cuts up the entire height of the fundamental domain. We will not consider this possibility for reasons that will become clear shortly.

The overriding goal of this calculation will be to show how to recover beta functions for the Yukawa couplings in the field theory limit. We will therefore mainly be interested in the limit of large t (where t is the ratio of the annulus length to its width, and plays the role of the Schwinger time) and in particular the dependence of the results on the IR (large-t) cut-off. In this limit the beta functions are dominated by the various field theory limits in which one or more of the vertices are pinched together. The relevant diagrams for discussing beta functions will be those that factor on a Yukawa coupling times field renormalisation diagram. There are four different limiting cases shown in figure 2, which we will refer to as limits 0 to 4. Limit 0 is the partition function factorisation limit where all the vertices come together, and the string diagram factorises into the product of a one-loop annulus diagram and the tree-level Yukawa coupling.

Adjacent to the other diagrams are their nearest field theory equivalents. Concentrating on the quantum part of the amplitude for a moment, limit 1 (non-degenerate vertices) is a coupling renormalisation diagram and so in supersymmetric theories this 'limit' should give zero. In $\mathcal{N} = 0$ models such diagrams will be non-zero and will represent an actual coupling renormalisation. (However, the $t \to 0$ limit would yield UV divergences in these cases due to non-vanishing tadpoles in the closed string channel, indicating a non-trivial background.) One of our tasks therefore will be to show the vanishing of this contribution in supersymmetric configurations due to a ϑ-function identity. This is the stringy version of the non-renormalisation theorem. The only opportunity to obtain non-zero contributions is therefore when there is a pole, corresponding to limits 2 and 3.

One diagram that we will not consider is the the one which would be a Yukawa renormalisation with three intersection (twisted) fields in the loop. It is this case which corresponds to the diagram with a non-parallel fourth brane and branch-cut all the way up. Calculating them would involve a significant complication, but since they can only be relevant for non-supersymmetric theories anyway, we feel justified in neglecting them.

Our expectations for the classical worldsheet-instanton contributions to the amplitudes are based on the quantum part. Field theoretically the closest diagram to limit 1 would be proportional to $\mathcal{Y}g^2$ where $\mathcal{Y}$ is the Yukawa coupling and g the relevant gauge coupling of the gauge field in the loop. The Yukawa coupling at tree level is determined by the worldsheet instanton whose action is simply the sum of the projected triangle areas swept

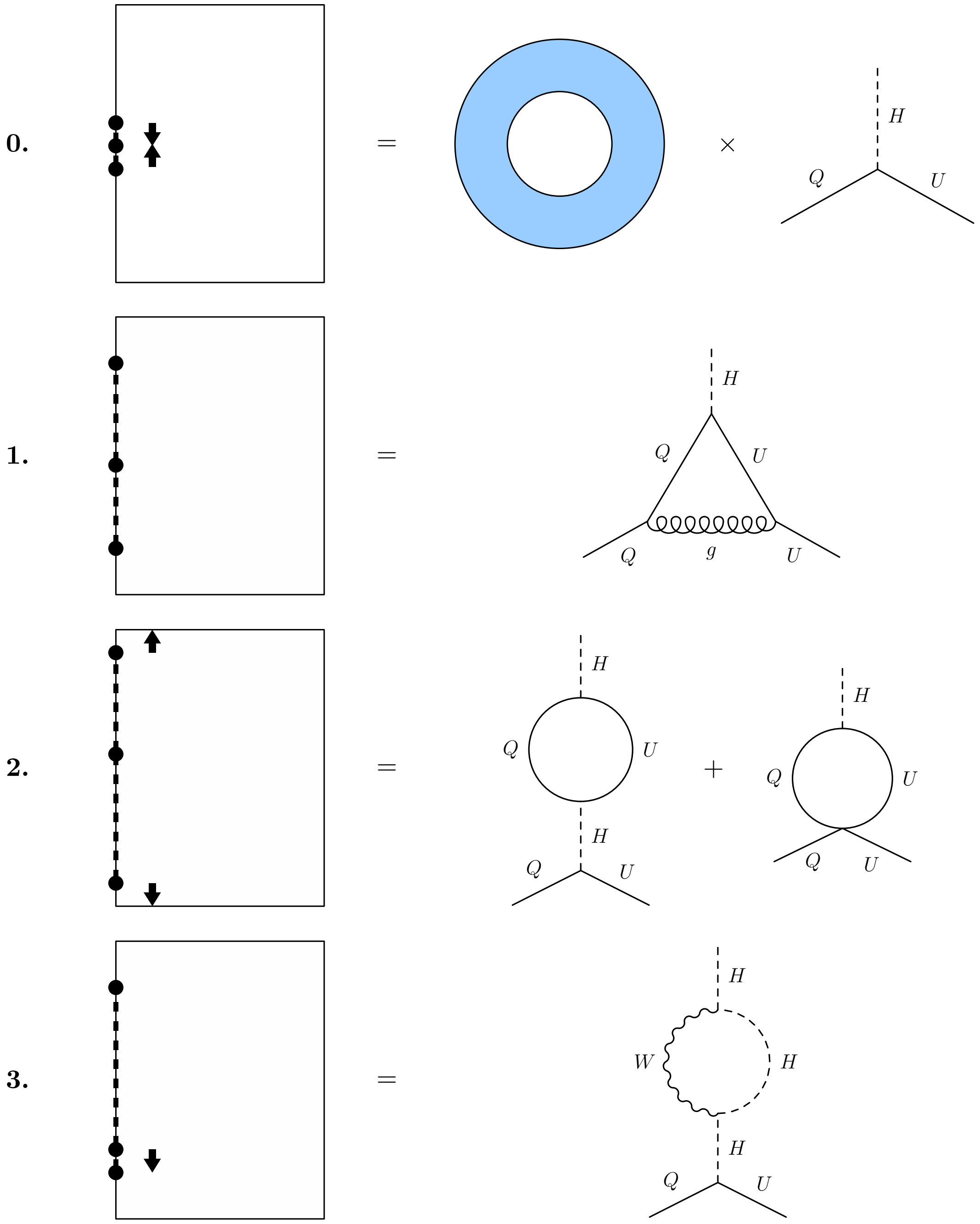

Figure 2: Field-theory diagrams from various limits on the worldsheet.

out by the worldsheet in the three sub-tori, $S_{\text{tree}} = \sum_{i=1}^{3} (\text{Area})_i$ (the tree-level Yukawa is proportional to $e^{-S_{\text{tree}}}$). The same sum of triangle areas should appear in the one-loop action in this limit.

Limit 2 gives the field theory diagram corresponding to a Yukawa contribution with a bubble on one of the legs, which has two twisted states running in the loop. We can see this heuristically by noting that the imaginary direction on the annulus represents the loop, and that the branch-cut free part of the annulus is pinched in this limit. These contributions should be proportional to $\mathcal{Y}\mathcal{Y}^\dagger\mathcal{Y}$. This means that the one-loop classical action should yield $S_{\text{one-loop}} = \sum S_{\text{tree}}$ in this limit, where the sum is over Yukawas appearing in the field theory diagram. The diagram should also have a $1/k^2$ pole that requires the k^2 term in the quantum prefactor; this is simply the contribution to the Yukawa beta function coming from field renormalisation. As all states are twisted, there should be no Kaluza-Klein (KK) modes in this limit.

Limit 3 gives the field theory diagram with one twisted state and one untwisted (gauge) state in the loop. These contributions should still be proportional to $\mathcal{Y}g^2$. Note that we should get contributions from all the KK modes of the gauge field, generating power-law running of the Yukawa coupling in this limit. This sum over KK contributions comes from a Poisson resummation of the classical action contribution to the amplitude.

The paper is organised as follows. In the next section we set up the string theory calculation, discussing vertex operators and charge conservation, and extract the general form of the amplitudes in terms of correlators. We then evaluate all necessary correlators, including the spin and twist fields, on the annulus. In section 3, we collect the necessary correlators together and discuss the quantum part of the amplitude, elucidating the emergence of the non-renormalisation theorem. Section 4 is devoted to deriving the classical instanton action via monodromy conditions. Section 5 discusses the factorisation of the classical part onto the various limits discussed above, while section 6 discusses the extraction of the beta functions and power law running.

2. Elements of the calculation

Before we begin, it is perhaps worthwhile to give a brief review of background literature. The supersymmetric non-renormalisation theorem has been demonstrated explicitly for a $\mathbb{Z}_3$ orbifold using only untwisted states [22], and more general rules have also been derived for other $D = 4$ models [23]. Also, an extraction of the β-function for gauge fields was performed for supersymmetric orbifolds [24].

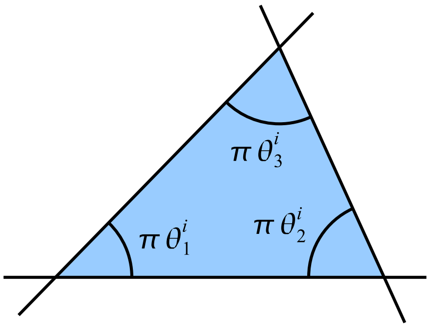

Figure 3: We work with internal angles, and always stick to the notation where subscripts on angles label intersections, and superscripts label tori.

Key elements in our calculation will be the conservation of the H-charge of bosonised states, correlation functions of spin operators and correlation functions of twist operators. H-charge conservation and consequent selection rules are discussed for orbifold models in [25 – 31]. Spin field operators for closed strings at one loop have been given explicitly for flat backgrounds [32 – 34], and implicitly for orbifold backgrounds [35]; we give an explicit result and generalise to open strings in section 2.4. Similarly, twist field correlators are known at one loop for closed strings [36], which we generalise to open strings in section 2.6. We also examine correlation functions of excited twist operators at one loop, which have been discussed for the orbifold case in [37 – 39].

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We will work with type-IIA theory compactified on a factorisable T^6, with D6-branes at angles wrapping in the compact space. In each sub-torus, the intersection of the branes makes a triangle. The rotations which take us from one brane to another are taken as shown: hence, $\theta_1^i + \theta_2^i + \theta_3^i = 1$ in each sub-torus. (This choice differs from the bulk of the literature in which $\sum_j \theta_j^i = 2$ in each sub-torus: the replacement $\theta \to 1 - \theta$ relates the two conventions). Unless displayed explicitly, we set $\alpha' = \frac{1}{2}$.

2.1 Vertex operators and H-charge

At any given intersection, an analysis of the mode expansions of strings stretched between the branes [40] tells us that we obtain from the R sector a massless fermion. Introducing a set H_i of bosonic fields on the worldsheet, where i labels pairs of complex dimensions, the most canonical vertex operator for this state is [41 − 43]

$$\mathcal{V}_{-\frac{1}{2}}\left(u, k, z\right) = \mathrm{e}^{-\frac{1}{2}\phi} u S^{\pm} \mathrm{e}^{\mathrm{i}k\cdot X} \mathrm{e}^{\mathrm{i}q\cdot H} \Sigma\left(z\right) \qquad \text{with} \qquad q = \left(\frac{1}{2} - \theta^1, \frac{1}{2} - \theta^2, \frac{1}{2} - \theta^3\right), \tag{2.1}$$

where u is the four-dimensional polarization, $S^{\pm} = \mathrm{e}^{\pm\frac{1}{2}\mathrm{i}(H_1+H_2)}$ is a four-dimensional spin field, $\mathrm{e}^{\mathrm{i}q\cdot H}$ is the spin field for the internal co-ordinates, and $\Sigma = \sigma^1\sigma^2\sigma^3$ are bosonic twist fields [37, 44]. Our convention for angles means that the operators σ correspond to 'anti-twist' fields, sometimes denoted σ_- or $\sigma_{1-\theta}$ in the literature. The plane wave portion of the vertex operators, $\mathrm{e}^{\mathrm{i}k\cdot X}$, is present only in the non-compact space — k^μ is a four-vector. We have suppressed factors of the string coupling g_O and Chan-Paton factors in this expression, and will do so throughout the rest of this work.

A vertex operator of the opposite chirality may also be created, representing strings stretched with the opposite orientation, but as our final result ought to be comparable to a quark Yukawa term, in which both fermions have the same chirality, we will not need it.

In the NS sector, the scalars surviving the GSO projection are generally massive [40],

State	q	m^2
$\Psi_{\frac{1}{2}-\theta^1}\left\|0\right\rangle$	$\left(-\theta^1, 1-\theta^2, 1-\theta^3\right)$	$\theta^1 - \theta^2 - \theta^3 + 1$
$\Psi_{\frac{1}{2}-\theta^2}\left\|0\right\rangle$	$\left(1-\theta^1, -\theta^2, 1-\theta^3\right)$	$-\theta^1 + \theta^2 - \theta^3 + 1$
$\Psi_{\frac{1}{2}-\theta^3}\left\|0\right\rangle$	$\left(1-\theta^1, 1-\theta^2, -\theta^3\right)$	$-\theta^1 - \theta^2 + \theta^3 + 1$
$\Psi_{\frac{1}{2}-\theta^1}\Psi_{\frac{1}{2}-\theta^2}\Psi_{\frac{1}{2}-\theta^3}\left\|0\right\rangle$	$\left(-\theta^1, -\theta^2, -\theta^3\right)$	$\theta^1 + \theta^2 + \theta^3 - 1$

With a suitable choice of angles, we may make any one (but only one) of these states massless, in which case it becomes the $\mathcal{N} = 1$ superpartner of the massless fermion in the R sector. The other three states are then massive, and so must be superpartners of other heavy fermions. It may appear strange that we have a choice of four NS-sector vertex operators but only one R-sector vertex operator. In fact, up to chirality, there are *four* choices which could have been made for (2.1), the other three being obtained by flipping the sign of the $\frac{1}{2}$ in any pair of sub-tori.

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For the choice shown in (2.1), we may identify the Higgs from factorisation of the tree level four-point function [21] as the state $\Psi_{\frac{1}{2}-\theta^1}\Psi_{\frac{1}{2}-\theta^2}\Psi_{\frac{1}{2}-\theta^3}\left|0\right\rangle$. The $\mathcal{N}=1$ supersymmetry condition on our set-up is then

$$\theta^1+\theta^2+\theta^3=1\,, \tag{2.2}$$

and the most canonical vertex operator for the Higgs is

$$\mathcal{V}_{-1}\left(k,z\right)=\mathrm{e}^{-\phi}\mathrm{e}^{\mathrm{i}k\cdot X}\mathrm{e}^{\mathrm{i}q\cdot H}\Sigma\left(z\right)\qquad\text{with}\quad q=\left(-\theta^1,-\theta^2,-\theta^3\right). \tag{2.3}$$

One may verify that (2.3) correctly has unit conformal weight, provided that $k^2=-m^2$.

Another way to see that that we have identified the correct state as the Higgs is to note that since the H are a set of free bosonic fields, non-vanishing terms in $\mathcal{A}$ must obey H-charge (momentum) conservation:

$$\sum_{i=1}^{3}q_i^j=0 \tag{2.4}$$

in each complex dimension j. If we were interested in calculating the *tree-level* Yukawa coupling,

$$\mathcal{Y}=\left\langle\mathcal{V}_{-\frac{1}{2}}\mathcal{V}_{-\frac{1}{2}}\mathcal{V}_{-1}\right\rangle, \tag{2.5}$$

we would see that H-charge conservation is correctly obeyed by the vertex operators given: in an external dimension, $\pm\frac{1}{2}\mp\frac{1}{2}=0$, and in an internal dimension $\frac{1}{2}-\theta_1+\frac{1}{2}-\theta_2-\theta_3=0$.

Using the simple tool of H-charge conservation, the reader may verify that choices for the quantum numbers of the fermion vertex operator other than those displayed in (2.1) will lead to the Higgs being identifed as one of the other three NS-sector states in the table; the $N=1$ supersymmetry condition on the angles will then be that which makes the appropriate state massless. For concreteness, we will stick to the choices (2.1) and (2.3).

2.2 One-loop amplitudes and picture-changing

On the annulus, the appropriate string scattering amplitude is given by finding the correlation function of two fermions plus one boson, then integrating over all possible configurations of vertex operators plus the modular parameter t of the annulus:

$$\mathcal{A}=\int_0^\infty\frac{\mathrm{d}t}{t}f\left(t\right)\int\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3\,A\left(z_1,z_2,z_3\right). \tag{2.6}$$

$f\left(t\right)$ is an overall normalisation which will be determined later by coalescing the three vertex operators (limit 0 of figure 2). For now, we focus on the computation of A.

At tree-level, the amplitude (2.5) has an overall ϕ-charge of $-\frac{1}{2}-\frac{1}{2}-1=-2$. On the annulus, $\mathcal{A}$ must have an overall ϕ-charge of zero if it is to be non-anomalous, and we have to perform 'picture-changing' operations [45] on our vertex operators:

$$\mathcal{V}_{i+1}\left(z\right)=\lim_{w\to z}\mathrm{e}^{\phi}T_F\left(w\right)\mathcal{V}_i\left(z\right). \tag{2.7}$$

Here, T_F is the generator of worldsheet supersymmetry,

$$T_F = \frac{1}{2}\left(\partial\overline{X}^i\psi_i + \partial X^i\overline{\psi}_i\right), \tag{2.8}$$

with $\psi \sim \mathrm{e}^{\mathrm{i}H}$ and $\overline{\psi} \sim \mathrm{e}^{-\mathrm{i}H}$ bosonised fermions in each complex pair of dimensions,

$$X^i = X^{\mu=2i} + \mathrm{i}X^{\mu=2i+1} \qquad \overline{X}^i = X^{\mu=2i} - \mathrm{i}X^{\mu=2i+1}\,. \tag{2.9}$$

To obtain the correct ϕ-charge, we need to insert two picture-changing operators somewhere in our amplitude. It will be convenient to apply one operator to one of the fermions, and one to the boson, so that the relevant correlation function is

$$\mathcal{A} = \left\langle \mathcal{V}_{-\frac{1}{2}}\left(\overline{u}_1, k_1, z_1\right) \lim_{w_2\to z_2} \mathrm{e}^{\phi}T_F\left(w_2\right)\mathcal{V}_{-\frac{1}{2}}\left(u_2, k_2, z_2\right) \lim_{w_3\to z_3} \mathrm{e}^{\phi}T_F\left(w_3\right)\mathcal{V}_{-1}\left(u_3, k_3, z_3\right)\right\rangle. \tag{2.10}$$

Our picture-changing operators must be inserted in such a way as not to affect the overall H-charge, meaning that for each complex dimension, we should make one $\partial\overline{X}\psi$ and one $\partial X\overline{\psi}$ insertion. In other words, only terms of the form

$$\mathcal{V}_{-\frac{1}{2}}\left(z_1\right)\partial\overline{X}^i\psi^i\left(w_2\right)\mathcal{V}_{-\frac{1}{2}}\left(z_2\right)\partial X^i\overline{\psi}^i\left(w_3\right)\mathcal{V}_{-1}\left(z_3\right) \tag{2.11}$$

and

$$\mathcal{V}_{-\frac{1}{2}}\left(z_1\right)\partial X^i\overline{\psi}^i\left(w_2\right)\mathcal{V}_{-\frac{1}{2}}\left(z_2\right)\partial\overline{X}^i\psi^i\left(w_3\right)\mathcal{V}_{-1}\left(z_3\right) \tag{2.12}$$

may contribute to A (no summation over i is implied).

We may see explicitly the effect of the picture-changing operators on the vertex operators (2.1) and (2.3) by using the OPEs

$$\begin{aligned}
\mathrm{e}^{iaH}\left(w\right)\mathrm{e}^{ibH}\left(z\right) &\sim \left(w-z\right)^{ab}\mathrm{e}^{i(a+b)H}\left(z\right)\\
\partial X\left(w\right)\mathrm{e}^{ik\cdot X}\left(z\right) &\sim \frac{\mathrm{i}k}{w-z}\mathrm{e}^{ik\cdot X}\left(z\right)\\
\partial\overline{X}\left(w\right)\mathrm{e}^{ik\cdot X}\left(z\right) &\sim \frac{\mathrm{i}\overline{k}}{w-z}\mathrm{e}^{ik\cdot X}\left(z\right),
\end{aligned} \tag{2.13}$$

where k^i is defined in exactly the same way as X^i, so that $k\cdot X = \frac{1}{2}\left(k^i\overline{X}^i + \overline{k}^iX^i\right)$, and

$$\begin{aligned}
\partial X\left(w\right)\sigma\left(z\right) &\sim \left(w-z\right)^{-\theta}\tau\left(z\right)\\
\partial\overline{X}\left(w\right)\sigma\left(z\right) &\sim \left(w-z\right)^{-(1-\theta)}\tau'\left(z\right),
\end{aligned} \tag{2.14}$$

with τ and τ' excited twist fields.[1]

Since the vertex operators are segregated into operators which act only in the internal dimensions ($S^{\pm}$, $\mathrm{e}^{\mathrm{i}k\cdot X}$) and operators which act only in the external dimensions ($\mathrm{e}^{\mathrm{i}q\cdot H}$, Σ), we need to treat internal and external indices differently. Beginning with the case where i is an internal index, we have for the fermion,

$$\frac{1}{2}\lim_{w\to z}\mathrm{e}^{\phi}\partial\overline{X}^i\psi^i\left(w\right)\mathcal{V}_{-\frac{1}{2}}\left(z\right) = \frac{1}{2}\mathrm{e}^{\frac{1}{2}\phi}uS^{\pm}\mathrm{e}^{\mathrm{i}k\cdot X}\mathrm{e}^{\mathrm{i}\tilde{q}\cdot H}\tau'^i\sigma^j\sigma^k\left(z\right) \qquad \text{with } \tilde{q}^i = \frac{3}{2}-\theta^i \tag{2.15}$$

[1] We have made the replacement $\theta \to 1-\theta$ with respect to the usual definition of these OPEs, reflecting the fact that we work with internal angles as shown in figure 3.

Sub-torus / H-charge	i	j	k
q_1	$\frac{1}{2}-\theta_1^i$	$\frac{1}{2}-\theta_1^j$	$\frac{1}{2}-\theta_1^k$
$\tilde{q}_2$	$-\frac{1}{2}-\theta_2^i$	$\frac{1}{2}-\theta_2^j$	$\frac{1}{2}-\theta_2^k$
$\tilde{q}_3$	$1-\theta_3^i$	$-\theta_3^j$	$-\theta_3^k$

Table 1: H-charge assignments in the contribution $\mathcal{A}_1$. The sum in each column is zero, preserving charge conservation in each sub-torus.

$$\frac{1}{2}\lim_{w\to z} \mathrm{e}^{\phi}\partial X^i\overline{\psi}^i\left(w\right)\mathcal{V}_{-\frac{1}{2}}\left(z\right) = \frac{1}{2}\mathrm{e}^{\frac{1}{2}\phi}uS^{\pm}\mathrm{e}^{\mathrm{i}k\cdot X}\mathrm{e}^{\mathrm{i}\tilde{q}\cdot H}\tau^i\sigma^j\sigma^k\left(z\right) \qquad \text{with } \tilde{q}^i = -\frac{1}{2}-\theta^i\,,$$

again with no summation implied over i.

For the boson,

$$\begin{aligned}\frac{1}{2}\lim_{w\to z} \mathrm{e}^{\phi}\partial\overline{X}^i\psi^i\left(w\right)\mathcal{V}_{-1}\left(z\right) &= \frac{1}{2}\mathrm{e}^{\mathrm{i}k\cdot X}\mathrm{e}^{\mathrm{i}\tilde{q}\cdot H}\tau'^i\sigma^j\sigma^k\left(z\right) \qquad \text{with} \quad \tilde{q}^i = 1-\theta^i\\ \frac{1}{2}\lim_{w\to z} \mathrm{e}^{\phi}\partial X^i\overline{\psi}^i\left(w\right)\mathcal{V}_{-1}\left(z\right) &= 0\,.\end{aligned} \tag{2.16}$$

Therefore, the internal dimensions contribute only to $\mathcal{A}$ via terms of the form (2.12). We label these contributions as A_1:

$$\begin{aligned}A_1 = {}& \frac{1}{4}\overline{u}_1u_2\left\langle \mathrm{e}^{-\frac{1}{2}\phi}\left(z_1\right)\mathrm{e}^{\frac{1}{2}\phi}\left(z_2\right)\right\rangle\left\langle S^{\mp}\left(z_1\right)S^{\pm}\left(z_2\right)\right\rangle\left\langle \mathrm{e}^{\mathrm{i}k_1\cdot X}\left(z_1\right)\mathrm{e}^{\mathrm{i}k_2\cdot X}\left(z_2\right)\mathrm{e}^{\mathrm{i}k_3\cdot X}\left(z_3\right)\right\rangle \times\\ &\times\sum_{i\neq j\neq k}\left\langle \mathrm{e}^{\mathrm{i}q_1\cdot H(z_1)}\mathrm{e}^{\mathrm{i}\tilde{q}_2\cdot H(z_2)}\mathrm{e}^{\mathrm{i}\tilde{q}_3\cdot H(z_3)}\right\rangle\left\langle \sigma^i(z_1)\tau^i(z_2)\tau'^i(z_3)\right\rangle\left\langle \sigma^j(z_1)\sigma^j(z_2)\sigma^j(z_3)\right\rangle \times\\ &\times\left\langle \sigma^k(z_1)\sigma^k(z_2)\sigma^k(z_3)\right\rangle.\end{aligned} \tag{2.17}$$

The arrangement of picture-changed H-charge in the spin operators here is displayed explicitly in table 1.

Now consider the case when i is an external index. There are two different possibilities for the result of the picture-changing operation on the fermion operator $\mathcal{V}_{-\frac{1}{2}}$, depending upon whether it contains an $S^+ = \mathrm{e}^{+\frac{1}{2}\mathrm{i}(H_1+H_2)}$ or an $S^- = \mathrm{e}^{-\frac{1}{2}\mathrm{i}(H_1+H_2)}$ operator. If $\mathcal{V}_{-\frac{1}{2}}$ contains an S^+, then we have

$$\begin{aligned}\frac{1}{2}\lim_{w\to z} \mathrm{e}^{\phi}\partial\overline{X}^i\psi^i\left(w\right)\mathcal{V}_{-\frac{1}{2}}\left(z\right) &= \frac{1}{2}\mathrm{i}k^i\mathrm{e}^{\frac{1}{2}\phi}\mathrm{e}^{\mathrm{i}k\cdot X}S^+_{\uparrow}\mathrm{e}^{\mathrm{i}q\cdot H}\Sigma\left(z\right)\\ \frac{1}{2}\lim_{w\to z} \mathrm{e}^{\phi}\partial X^i\overline{\psi}^i\left(w\right)\mathcal{V}_{-\frac{1}{2}}\left(z\right) &= \frac{1}{2}\partial X^i\mathrm{e}^{\frac{1}{2}\phi}\mathrm{e}^{\mathrm{i}k\cdot X}S^+_{\downarrow}\mathrm{e}^{\mathrm{i}q\cdot H}\Sigma\left(z\right) + \text{singular term}\end{aligned} \tag{2.18}$$

where

$$S^+_{\uparrow} = \mathrm{e}^{\mathrm{i}\left(\frac{3}{2}H_i+\frac{1}{2}H_j\right)} \qquad \text{and} \qquad S^+_{\downarrow} = \mathrm{e}^{\mathrm{i}\left(-\frac{1}{2}H_i+\frac{1}{2}H_j\right)}\,. \tag{2.19}$$

The singular term will vanish via the Dirac equation, just as it does in the picture-changing of $\mathcal{V}_{-\frac{1}{2}}$ when no branes are present [45, 46]; this vanishing has been obscured by the 'helicity basis' used to represent the spin operators.

If $\mathcal{V}_{-\frac{1}{2}}$ contains an S^- operator,

$$\begin{aligned}
\frac{1}{2}\lim_{w\to z} \mathrm{e}^{\phi}\partial\overline{X}^i\psi^i(w)\,\mathcal{V}_{-\frac{1}{2}}(z) &= \frac{1}{2}\partial\overline{X}^i\mathrm{e}^{\frac{1}{2}\phi}\mathrm{e}^{\mathrm{i}k\cdot X}S^-_{\uparrow}\mathrm{e}^{\mathrm{i}q\cdot H}\Sigma(z) + \text{singular term} \\
\frac{1}{2}\lim_{w\to z} \mathrm{e}^{\phi}\partial X^i\overline{\psi}^i(w)\,\mathcal{V}_{-\frac{1}{2}}(z) &= \frac{1}{2}\mathrm{i}k^i\mathrm{e}^{\frac{1}{2}\phi}\mathrm{e}^{\mathrm{i}k\cdot X}S^-_{\downarrow}\mathrm{e}^{\mathrm{i}q\cdot H}\Sigma(z)
\end{aligned}\tag{2.20}$$

where

$$S^-_{\uparrow} = \mathrm{e}^{\mathrm{i}\left(\frac{1}{2}H_i-\frac{1}{2}H_j\right)} \qquad \text{and} \qquad S^-_{\downarrow} = \mathrm{e}^{\mathrm{i}\left(-\frac{3}{2}H_i-\frac{1}{2}H_j\right)}. \tag{2.21}$$

For the boson,

$$\begin{aligned}
\lim_{w\to z} \mathrm{e}^{\phi}\partial\overline{X}^i\psi^i(w)\,\mathcal{V}_{-1}(z) &= \frac{1}{2}\mathrm{i}\overline{k}^i\psi^i\mathrm{e}^{\mathrm{i}k\cdot X}\mathrm{e}^{\mathrm{i}q\cdot H}\Sigma(z) \\
\lim_{w\to z} \mathrm{e}^{\phi}\partial X^i\overline{\psi}^i(w)\,\mathcal{V}_{-1}(z) &= \frac{1}{2}\mathrm{i}k^i\overline{\psi}^i\mathrm{e}^{\mathrm{i}k\cdot X}\mathrm{e}^{\mathrm{i}q\cdot H}\Sigma(z).
\end{aligned}\tag{2.22}$$

Since we should consider both possibilities for the boson, there are four possible terms from picture-changing the external dimensions: two from the H-charge conserving combinations of (2.18) and (2.22), and two from the H-charge conserving combinations of (2.20) and (2.22). The terms naturally group into two pairs:

$$\begin{aligned}
A_2^+ &= \frac{1}{4}\overline{u}_1u_2\mathcal{I}\left\langle \mathrm{e}^{-\frac{1}{2}\phi}(z_1)\,\mathrm{e}^{\frac{1}{2}\phi}(z_2)\right\rangle \sum_{i\neq j}^{2}\mathrm{i}\overline{k}_3^i\left\langle \mathrm{e}^{\mathrm{i}k_1\cdot X}(z_1)\,\partial X^i\mathrm{e}^{\mathrm{i}k_2\cdot X}(z_2)\,\mathrm{e}^{\mathrm{i}k_3\cdot X}(z_3)\right\rangle \times \\
&\quad\times\left\langle \mathrm{e}^{-\frac{1}{2}\mathrm{i}H_i}(z_1)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}H_i}(z_2)\,\mathrm{e}^{\mathrm{i}H_i}(z_3)\right\rangle\left\langle \mathrm{e}^{-\frac{1}{2}\mathrm{i}H_j}(z_1)\,\mathrm{e}^{\frac{1}{2}\mathrm{i}H_j}(z_2)\right\rangle
\end{aligned}\tag{2.23}$$

$$\begin{aligned}
A_2^- &= \frac{1}{4}\overline{u}_1u_2\mathcal{I}\left\langle \mathrm{e}^{-\frac{1}{2}\phi}(z_1)\,\mathrm{e}^{\frac{1}{2}\phi}(z_2)\right\rangle \sum_{i\neq j}^{2}\mathrm{i}k_3^i\left\langle \mathrm{e}^{\mathrm{i}k_1\cdot X}(z_1)\,\partial X^i\mathrm{e}^{\mathrm{i}k_2\cdot X}(z_2)\,\mathrm{e}^{\mathrm{i}k_3\cdot X}(z_3)\right\rangle \times \\
&\quad\times\left\langle \mathrm{e}^{\frac{1}{2}\mathrm{i}H_i}(z_1)\,\mathrm{e}^{\frac{1}{2}\mathrm{i}H_i}(z_2)\,\mathrm{e}^{-\mathrm{i}H_i}(z_3)\right\rangle\left\langle \mathrm{e}^{\frac{1}{2}\mathrm{i}H_j}(z_1)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}H_j}(z_2)\right\rangle,
\end{aligned}\tag{2.24}$$

and

$$\begin{aligned}
A_3^+ &= -\frac{1}{4}\overline{u}_1u_2\mathcal{I}\left\langle \mathrm{e}^{-\frac{1}{2}\phi}(z_1)\,\mathrm{e}^{\frac{1}{2}\phi}(z_2)\right\rangle\left\langle \mathrm{e}^{\mathrm{i}k_1\cdot X}(z_1)\,\mathrm{e}^{\mathrm{i}k_2\cdot X}(z_2)\,\mathrm{e}^{\mathrm{i}k_3\cdot X}(z_3)\right\rangle \times \\
&\quad\times\sum_{i\neq j}^{2}k_2^i\overline{k}_3^i\left\langle \mathrm{e}^{-\frac{1}{2}\mathrm{i}H_i}(z_1)\,\mathrm{e}^{\frac{3}{2}\mathrm{i}H_i}(z_2)\,\mathrm{e}^{-\mathrm{i}H_i}(z_3)\right\rangle\left\langle \mathrm{e}^{-\frac{1}{2}\mathrm{i}H_j}(z_1)\,\mathrm{e}^{\frac{1}{2}\mathrm{i}H_j}(z_2)\right\rangle
\end{aligned}\tag{2.25}$$

$$\begin{aligned}
A_3^- &= -\frac{1}{4}\overline{u}_1u_2\mathcal{I}\left\langle \mathrm{e}^{-\frac{1}{2}\phi}(z_1)\,\mathrm{e}^{\frac{1}{2}\phi}(z_2)\right\rangle\left\langle \mathrm{e}^{\mathrm{i}k_1\cdot X}(z_1)\,\mathrm{e}^{\mathrm{i}k_2\cdot X}(z_2)\,\mathrm{e}^{\mathrm{i}k_3\cdot X}(z_3)\right\rangle \times \\
&\quad\times\sum_{i\neq j}^{2}\overline{k}_2^ik_3^i\left\langle \mathrm{e}^{\frac{1}{2}\mathrm{i}H_i}(z_1)\,\mathrm{e}^{-\frac{3}{2}\mathrm{i}H_i}(z_2)\,\mathrm{e}^{\mathrm{i}H_i}(z_3)\right\rangle\left\langle \mathrm{e}^{\frac{1}{2}\mathrm{i}H_j}(z_1)\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}H_j}(z_2)\right\rangle,
\end{aligned}\tag{2.26}$$

where the contribution from the internal dimensions is

$$\mathcal{I} = \left\langle \mathrm{e}^{\mathrm{i}q_1\cdot H}(z_1)\,\mathrm{e}^{\mathrm{i}q_2\cdot H}(z_2)\,\mathrm{e}^{\mathrm{i}q_3\cdot H}(z_3)\right\rangle \langle \Sigma_1(z_1)\,\Sigma_2(z_2)\,\Sigma_3(z_3)\rangle\,. \tag{2.27}$$

So, overall, we have five different terms contributing to A. Heuristically, the difference between them is that A_1 contains excited twist operators but contains no kinematic factors, whereas the $A_2^{\pm}$ and $A_3^{\pm}$ terms contain only ordinary twist operators but have kinematic factors in front of them. These kinematic factors will generally cause the $A_2^{\pm}$ and $A_3^{\pm}$ terms to be suppressed relative to the A_1 contribution, unless we bring two of the vertex operators close together. Therefore, it seems appropriate to identify the term A_1 with limit 1 in figure 2, and $A_2^{\pm}$, $A_3^{\pm}$ with the other cases [22]. When we compute these terms explicitly, we will see that this identification is indeed correct; first, however, we must find explicit expressions for the correlation functions in A_1, A_2 and A_3. The discussion is somewhat technical, so for reference we have collected the results at the beginning of the section 3, where we start to build the amplitudes.

2.3 Bosonic fields

Defining the annulus as the rectangular region of the complex plane $x \in \left[0, \frac{1}{2}\right]$, $y \in [0,t]$, the correlators involving the fields X^{μ} may be found from those on the torus via the method of images [47]. The result is

$$\begin{aligned} \langle X(z_1)\,X(z_2)\rangle &= -\frac{1}{2}\left|\log\vartheta_1\left(z_2 - z_1|\mathrm{i}t\right)\right| + \frac{\pi\left[\mathrm{Im}(z_2 - z_1)\right]^2}{2t} - \\ &\quad -\frac{1}{2}\left|\log\vartheta_1\left(z_2 + \overline{z}_1|\mathrm{i}t\right)\right| + \frac{\pi\left[\mathrm{Im}(z_2 + \overline{z}_1)\right]^2}{2t} + r(t)\,, \end{aligned} \tag{2.28}$$

with the Jacobi theta function ϑ_1 defined as in appendix A, and $r(t)$ a suitable function which keeps the propagator regular as $z_1 \to z_2$. All of our fields are on the same end of the annulus, so we can set $z_1 = \mathrm{i}y_1$, $z_2 = \mathrm{i}y_2$ with $y_1, y_2 \in [0,t]$. Then, since the theta functions are real for purely imaginary arguments,

$$\langle X(y_1)\,X(y_2)\rangle = -\log\vartheta_1(\mathrm{i}y_{12}) + \frac{\pi y_{ij}^2}{t}\,, \tag{2.29}$$

with the notation $y_{ij} = |y_j - y_i|$. For brevity, we have suppressed the explicit t-dependence in ϑ_1. We have also left out the function $r(t)$; it will not become important until we begin to care about the t-dependence of A in section 6, and we will deal with it then.

With this simplification, the $\mathrm{e}^{\mathrm{i}k\cdot X}$ correlations in A are

$$\left\langle \prod_i \mathrm{e}^{\mathrm{i}k_i\cdot X}(y_i)\right\rangle = \prod_{i<j}\left[\vartheta_1(\mathrm{i}y_{ij})\,\mathrm{e}^{-\pi y_{ij}^2/t}\right]^{k_i\cdot k_j}\,, \tag{2.30}$$

while including one factor of ∂X^{μ} pulls down kinematic factors:

$$\left\langle \partial X^{\mu}(y_k)\prod_i \mathrm{e}^{\mathrm{i}k_i\cdot X}(y_i)\right\rangle = \prod_{\substack{i<j\\ i\neq k}} \mathrm{i}k_j^{\mu}\left[\frac{\theta_1'(\mathrm{i}y_{jk})}{\theta_1(\mathrm{i}y_{jk})} - \frac{2\pi y_{jk}}{t}\right]\left[\vartheta_1(\mathrm{i}y_{ij})\,\mathrm{e}^{-\pi y_{ij}^2/t}\right]^{k_i\cdot k_j}\,. \tag{2.31}$$

2.4 Spin fields

The correlators between the the fermion spin fields $\mathrm{e}^{\mathrm{i}q\cdot H}$ are more difficult to compute. The problem is that we must respect spin structures on the underlying torus, which prevents us from dealing with the H fields in the same way as the X fields. We may resolve the issue by using the stress-tensor method [37], generalising the results of ref. [34].

As above, we first perform the calculation on a torus, and then specialise the result to the annulus. Begin by bosonising the holomorphic fermions,

$$\psi(z) = \mathrm{e}^{\mathrm{i}H(z)} \qquad \overline{\psi}(z) = \mathrm{e}^{-\mathrm{i}H(z)}, \tag{2.32}$$

and defining holomorphic spin operators,

$$\mathcal{S}_a(z) = \mathrm{e}^{\mathrm{i}aH(z)}. \tag{2.33}$$

From (2.13), we see the the OPEs between fermions and spin fields take the form

$$\begin{aligned}
\psi(z)\,\overline{\psi}(w) &\sim (z-w)^{-1} \\
\psi(z)\,\mathcal{S}_a(w) &\sim (z-w)^{a}\;\mathcal{S}_{a+1}(w) \\
\overline{\psi}(z)\,\mathcal{S}_a(w) &\sim (z-w)^{-a}\;\mathcal{S}_{a-1}(w) \\
\mathcal{S}_a(z)\,\mathcal{S}_b(w) &\sim (z-w)^{ab}\;\mathcal{S}_{a+b}(w).
\end{aligned} \tag{2.34}$$

We will work out the correlation between an arbitrary number of spin fields, since it is no more complicated than that between three such fields. Define an auxiliary Green's function as

$$g(z,w;z_i) = \frac{\langle \overline{\psi}(z)\,\psi(w)\prod_i S_{a_i}(z_i)\rangle}{\langle \prod_i S_{a_i}(z_i)\rangle}. \tag{2.35}$$

The OPEs (2.34) show that this function must satisfy

$$\begin{aligned}
g(z,w;z_i) &= (z-w)^{-1} + \text{finite as } z\to w \\
g(z,w;z_i) &\propto (z-w)^{-a_i} + \text{finite as } z\to z_i \\
g(z,w;z_i) &\propto (z-w)^{a_i} + \text{finite as } w\to z_i\,.
\end{aligned} \tag{2.36}$$

Furthermore, it must be periodic on the torus as a function of z and w. A suitable function satisfying these conditions is

$$g(z,w;z_i) = \frac{\vartheta_1'(0)}{\vartheta_1(z-w)}\,\frac{\vartheta_{\alpha\beta}(z-w+\sum a_i z_i)}{\vartheta_{\alpha\beta}(\sum a_i z_i)}\prod_i\left(\frac{\vartheta_1(w-z_i)}{\vartheta_1(z-z_i)}\right)^{a_i}, \tag{2.37}$$

where $(\alpha\beta) = (00)\ldots(11)$ label the four possible spin structures on the torus. The reason for the specific form of the term $\sum_{i=1}^n a_i z_i$ in the argument of $\vartheta_{\alpha\beta}$ will become clear shortly; for the moment we press on.

The stress-energy tensor for the ψ, $\overline{\psi}$ CFT may be written as

$$T(z) = \lim_{z\to w}\left[\frac{1}{2}\partial_z\overline{\psi}(z)\,\psi(w) - \frac{1}{2}\overline{\psi}(z)\,\partial_w\psi(w) + \frac{1}{(z-w)^2}\right], \tag{2.38}$$

and so using the definition (2.35), we may form the function

$$\frac{\langle T(z)\prod_i S_{a_i}(z_i)\rangle}{\langle\prod_i S_{a_i}(z_i)\rangle} = \lim_{z\to w}\left[\frac{1}{2}\partial_z g(z,w;z_i) - \frac{1}{2}\partial_w g(z,w;z_i) + \frac{1}{(z-w)^2}\right]. \tag{2.39}$$

Using $\lim_{z\to 0}\vartheta_1(z) = \vartheta_1'(0)\,z$, and derivatives thereof, one finds

$$\frac{\langle T(z)\prod_i S_{a_i}(z_i)\rangle}{\langle\prod_i S_{a_i}(z_i)\rangle} = \frac{1}{2}\left[\sum_{i=1}^{n} a_i\frac{\vartheta_1'(z-z_i)}{\vartheta_1(z-z_i)}\right]^2 - \left[\sum_{i=1}^{n} a_i\frac{\vartheta_1'(z-z_i)}{\vartheta_1(z-z_i)}\right]\frac{\vartheta_{\alpha\beta}'\left(\sum_i a_i z_i\right)}{\vartheta_{\alpha\beta}\left(\sum_i a_i z_i\right)}.$$

The next step is to take the limit $z\to z_j$ on both sides. In this limit, the right-hand side may be evaluated directly, whilst the left-hand side is simplified by the fact that the OPE of any tensor operator with the stress-energy tensor takes the prescribed form

$$T(z)\,\mathcal{S}_{a_j}(z_j) \sim \frac{h}{(z-z_j)^2}\mathcal{S}_{a_j}(z_j) + \frac{1}{z-z_j}\partial_{z_j}\mathcal{S}_{a_j}(z_j), \tag{2.40}$$

with h the conformal weight of the field $\mathcal{S}_{a_j}(z)$. The result is

$$\begin{aligned}\lim_{z\to z_j}&\left\{\frac{h}{(z-z_j)^2} + \frac{1}{z-z_j}\partial_{z_j}\log\left\langle\prod_i \mathcal{S}_{a_i}(z_i)\right\rangle\right\} = \\ &= \lim_{z\to z_j}\left\{\frac{\frac{1}{2}a_j^2}{(z-z_j)^2} + \frac{a_j}{z-z_j}\left[\left(\sum_{i\neq j} a_i\frac{\vartheta_1'(z-z_i)}{\vartheta_1(z-z_i)}\right) - \frac{\vartheta_{\alpha\beta}'\left(\sum_i a_i z_i\right)}{\vartheta_{\alpha\beta}\left(\sum_i a_i z_i\right)}\right]\right\}.\end{aligned} \tag{2.41}$$

Comparing coefficients in $(z-z_j)$ shows that $\mathcal{S}_{a_j}(z)$ correctly has conformal weight $\frac{1}{2}a_j^2$, and that

$$\partial_{z_j}\log\left\langle\prod_i \mathcal{S}_{a_i}(z_i)\right\rangle = a_j\left[\left(\sum_{i\neq j} a_i\frac{\vartheta_1'(z-z_i)}{\vartheta_1(z-z_i)}\right) - \frac{\vartheta_{\alpha\beta}'\left(\sum_i a_i z_i\right)}{\vartheta_{\alpha\beta}\left(\sum_i a_i z_i\right)}\right], \tag{2.42}$$

with solution

$$\left\langle\prod_i \mathcal{S}_{a_i}(z_i)\right\rangle = K_{\alpha\beta}\left[\prod_{i<j}\vartheta_1(z_i-z_j)^{a_i a_j}\right]\vartheta_{\alpha\beta}\left(\sum_i a_i z_i\right) \tag{2.43}$$

where $K_{\alpha\beta}$ is an overall normalisation.

The reason for the argument of $\vartheta_{\alpha\beta}$ in (2.37) to take the form it does is now revealed: when $a_i = \pm\frac{1}{2}$, the three translations $z_i\to z_i+1$, $z_i\to z_i+\tau$ and $z_i\to z_i+1+\tau$ transform a given theta function $\vartheta_{\alpha\beta}$ into the other three theta functions. Under the same translations, the correlator (2.43) for a given spin structure transforms into the correlator for one of the other spin structures (up to a phase), just as it should. For the internal spin fields, on the other hand, translational invariance implies $\sum_i a_i = 0$, which is just the H-charge conservation condition that we used in section 2.1.

We can convert our result for general spin operators on the torus to that for the annulus quite easily, by defining an open-string auxiliary Green's function through the method of images,

$$g_{\text{open}}(z,w;z_i) = g(z,w;z_i) + g(z,\bar{w};z_i), \tag{2.44}$$

and proceeding through with the calculation as before. The outcome is that we end up with two identical copies of equation (2.42); we see, then, that the correlator for spin fields on the annulus is just the same as that on the torus.

2.5 Ghost spin fields

Also required are correlators for the ghost spin fields, $\mathrm{e}^{a\phi}$. These may be calculated by an analogous method, but with the OPEs and stress tensor appropriately modified to account for the fact that the ghost fields ϕ inhabit the $(\beta,\,\gamma)$ rather than the $\left(\overline{\psi},\,\psi\right)$ CFT [33]. The result turns out to be the reciprocal of (2.43):

$$\left\langle \mathrm{e}^{a_i\phi}\left(z_i\right)\right\rangle = K^g_{\alpha\beta}\left[\prod_{i<j}\vartheta_1\left(z_i-z_j\right)^{-a_ia_j}\right]\vartheta^{-1}_{\alpha\beta}\left(\sum_i a_iz_i\right). \tag{2.45}$$

2.6 Twist fields

Twist fields correlators are also tricky, this time because they do not possess a natural interpretation as local operators on the worldsheet. Again, the resolution is to use the stress-tensor method, and again we begin on a torus before converting our result to the annulus.

On a torus, the twist fields have the OPEs (2.14), given by the local monodromy conditions for the twist fields. Things are complicated somewhat by the *global* monodromy conditions — that is, the behaviour of the X fields as they are transported around collections of twist fields. If we define a closed loop γ as a loop on the worldsheet as a loop enclosing twist operators whose twists sum to zero, then around such a loop,

$$\Delta_\gamma X = \oint_\gamma \mathrm{d}z\,\partial X + \oint_\gamma \mathrm{d}\bar{z}\,\overline{\partial}X = v_\gamma\,. \tag{2.46}$$

Here v_γ represents a consistent displacement on the network of branes. If the branes are on a compact space then the contours may generate a displacement that wraps a number of times around the compactified brane before returning to the same place. We may choose to split X into a 'classical' part and a 'quantum fluctuation' part,

$$X = X_{\mathrm{cl}} + X_{\mathrm{qu}}\,, \tag{2.47}$$

with the requirement that the quantum part be unchanged by transportation around a closed loop while the classical part takes care of the displacement,

$$\Delta_\gamma X_{\mathrm{qu}} = \oint_\gamma \mathrm{d}z\,\partial X_{\mathrm{qu}} + \oint_\gamma \mathrm{d}\bar{z}\,\overline{\partial}X_{\mathrm{qu}} = 0\,. \tag{2.48}$$

$$\Delta_\gamma X_{\mathrm{cl}} = \oint_\gamma \mathrm{d}z\,\partial X_{\mathrm{cl}} + \oint_\gamma \mathrm{d}\bar{z}\,\overline{\partial}X_{\mathrm{cl}} = v_\gamma\,. \tag{2.49}$$

Then, the correlator for L twist operators also splits into a quantum and a classical part,

$$\left\langle \prod_{i=1}^{L}\sigma_i\left(z_i\right)\right\rangle = Z_\sigma \mathrm{e}^{-S_{\mathrm{cl}}}\,. \tag{2.50}$$

The classical portion of the twist correlators will play a key role in our calculation, and as such we defer discussion of it until section 4, considering first just the quantum part Z_σ.

Using the stress-tensor method to compute this correlator in a manner consistent with both the local and global monodromy conditions, plus the periodicity of the torus, is not a trivial task. Fortunately such a calculation has been performed on the torus by Atick et al. [36], with the result[2]

$$\begin{aligned} Z_\sigma = & \left|\det W\right|^{-1} \overline{\vartheta_1\left(Y\right)}^{(L-M-1)} \vartheta_1\left(Y'\right)^{(M-1)} \prod_{i<j}^{L-M} \overline{\vartheta_1\left(z_{\alpha_i}-z_{\alpha_j}\right)} \prod_{i<j}^{M} \vartheta_1\left(z_{\beta_i}-z_{\beta_j}\right) \times \\ & \times \prod_{i<j}^{L} \overline{\vartheta_1\left(z_i-z_j\right)}^{-(1-\theta_i)(1-\theta_j)} \vartheta_1\left(z_i-z_j\right)^{-\theta_i\theta_j} , \end{aligned} \tag{2.51}$$

where $M=\sum_{i=1}^{L}\theta_i$ is an integer, z_{α_i} are a set of $L-M$ twist insertion points chosen from z_i and z_{β_j} are a set of M twist insertion points chosen from z_i (not necessarily related to z_{α_i}). The elements of the elements of the $L\times L$ matrix W are given by

$$\begin{aligned} W_a^i &= \oint_{\gamma_a} \mathrm{d}z\, \omega_i\left(z\right) \qquad i=1,\ldots,L-M \\ W_a^i &= \oint_{\gamma_a} \mathrm{d}\bar{z}\, \overline{\omega_i'\left(\overline{z}\right)}, \qquad i=L-M+1,\ldots,L\,, \end{aligned} \tag{2.52}$$

with γ_a a basis for L closed loops on the worldsheet. ω and ω' are so-called 'cut differentials',

$$\begin{aligned} \omega_i\left(z\right) =& \gamma\left(z\right)\vartheta_1\left(z-z_{\alpha_i}-Y\right) \prod_{i\neq j}^{L-M} \vartheta_1\left(z-z_{\alpha_j}\right) \\ \omega_i'\left(z\right) =& \gamma'\left(z\right)\vartheta_1\left(z-z_{\beta_i}-Y'\right) \prod_{i\neq j}^{M} \vartheta_1\left(z-z_{\beta_j}\right) \end{aligned} \tag{2.53}$$

with the contributions

$$\begin{aligned} \gamma(z) &= \prod_{j=1}^{L} \vartheta_1\left(z-z_j\right)^{-(1-\theta_j)} \\ \gamma'(z) &= \prod_{j=1}^{L} \vartheta_1\left(z-z_j\right)^{-\theta_j} \end{aligned} \tag{2.54}$$

chosen to obey the local monodromy. These cut differentials form a basis for ∂X_{cl};

$$\partial \overline{X}_{\mathrm{cl}} = \sum_{i=1}^{L-M} c_i\omega_i$$

[2] The OPEs in [36] have $\theta\to 1-\theta$ with respect to our OPEs (2.14); in the conventional orbifold language, we are interested in the correlator of anti-twist operators, rather than twist operators. We deal with the disparity by reversing the notions of holomorphic and anti-holomorphic with respect to the work of Atick *et al.*, leading to the result shown in eq. (2.51). Formally, we should also replace $W\to\overline{W}$ everywhere; however, this does not affect our results, so we stick with the simpler notation.

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$$\partial X_{\rm cl} = \sum_{i=L-M+1}^{L} c_i \omega'_i\,. \tag{2.55}$$

The terms Y and Y' in (2.53) are

$$\begin{aligned} Y &= \sum_{i=1}^{L}(1-\theta_i)z_i - \sum_{j=1}^{L-M} z_{\alpha_j} \\ Y' &= \sum_{i=1}^{L}\theta_i z_i - \sum_{j=1}^{M} z_{\beta_j}\,, \end{aligned} \tag{2.56}$$

and have the function of keeping ω and ω' periodic on the torus.

This result may be converted to the annulus by applying the method of images, just as for the spin fields. This time, the two terms in the open-string Green's function lead to *different* differential equations; this is a reflection of the fact that the twist operators respond differently to holomorphic and anti-holomorphic fields on the torus, as we see from the OPEs above. Taking the vertex operators to lie along the annulus boundary, $z_i = \mathrm{i}y_i$, we find that both differential equations are in the same variable and so we may add them before solving. The result turns out to be the square root of (2.51), which is exactly what similar calculations at tree level [48, 21, 42] lead us to expect.

We are interested in the case where we have three twist operators, and the angles at the intersections add up to π; hence, $L = 3$ and $M = 1$. We choose the $3 - 1 = 2$ points z_{α_i} as $\{z_{\alpha_1}, z_{\alpha_2}\} = \{\mathrm{i}y_1, \mathrm{i}y_2\}$, and the one point $z_{\beta_1} = \mathrm{i}y_3$. Then, for a single complex dimension and up to an overall normalisation,

$$\begin{aligned} Z_\sigma &= |\det W|^{-\frac{1}{2}}\, \vartheta_1\left(\mathrm{i}\left(\theta_1 y_1 + \theta_2 y_2 - (1-\theta_3)y_3\right)\right)^{\frac{1}{2}} \vartheta_1\left(\mathrm{i}y_{12}\right)^{\frac{1}{2}} \times \\ &\quad \times \prod_{i<j}^{3} \vartheta_1\left(\mathrm{i}y_{ij}\right)^{-\frac{1}{2}(1-\theta_i)(1-\theta_j)-\frac{1}{2}\theta_i\theta_j} \end{aligned} \tag{2.57}$$

(an overall phase having been taken out of the first ϑ_1), with

$$\begin{aligned} W_\alpha^1 &= \oint_{\gamma_\alpha} \mathrm{d}z\, \gamma\left(z\right) \vartheta_1\left(z - \left(1-\theta_1\right)\mathrm{i}y_1 + \theta_2 \mathrm{i}y_2 - \left(1-\theta_3\right)y_3\right) \vartheta_1\left(z - \mathrm{i}y_2\right) \\ W_\alpha^2 &= \oint_{\gamma_\alpha} \mathrm{d}z\, \gamma\left(z\right) \vartheta_1\left(z + \theta_1 \mathrm{i}y_1 - \left(1-\theta_2\right)y_2 - \left(1-\theta_3\right)y_3\right) \vartheta_1\left(z - \mathrm{i}y_1\right) \\ W_\alpha^3 &= \oint_{\gamma_\alpha} \mathrm{d}\bar{z}\, \overline{\gamma'\left(\bar{z}\right)\vartheta_1\left(\bar{z} - \mathrm{i}\left(\theta_1 y_1 + \theta_2 y_2 + \theta_3 y_3\right)\right)}\,. \end{aligned} \tag{2.58}$$

Three suitable independent contours γ_a are shown in figure 4. They consist of a cycle γ_1 running around the annulus, a cycle γ_2 around the covering torus and a Pochhammer contour γ_3. This latter contour encircles each of the twist operators once in each direction, so that the net twist enclosed by γ_3 is correctly zero. We choose γ_3 to encircle the branch cut $\mathrm{i}y_1 < z < \mathrm{i}y_2$, but we could just have easily have chosen it to encircle the other cut; indeed, one may deform between the two by taking linear combinations of γ_3 together with the two cycles.

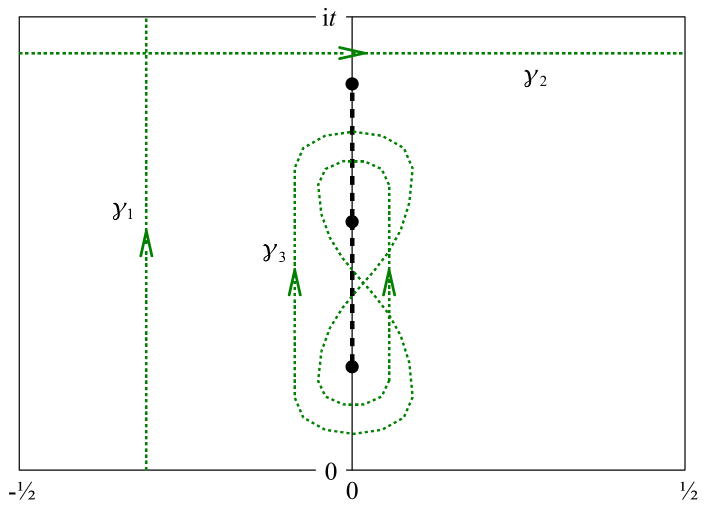

Figure 4: Independent contours on the worldsheet: two cycles and a Pochhammer loop. The annulus corresponds to the region $z > 0$.

Note that the cycle γ_2, which generates a displacement between the branes, is always chosen such that it does not pass through a branch cut. This corresponds to taking the displacement to be between two parallel pairs of branes.

2.7 Excited twist fields

Finally, we must deal with the excited twist correlator $\langle \sigma(z_1)\, \tau(z_2)\, \tau'(z_3)\rangle$ which is present in A_1 (eq. (2.17)). This correlator may be obtained from the basic twist correlator as follows: using the OPEs (2.14), write

$$\left\langle \sigma(z_1)\, \tau(z_2)\, \tau'(z_3)\right\rangle = \lim_{\substack{z\to z_2\\ w\to z_3}} (z-z_2)^{\theta_2}\,(w-z_3)^{1-\theta_3}\left\langle \sigma(z_1)\,\partial X(z)\,\sigma(z_2)\,\partial\overline{X}(w)\,\sigma(z_3)\right\rangle . \tag{2.59}$$

Writing $X = X_{\rm qu} + X_{\rm cl}$, this second correlator splits into a quantum and a classical piece,

$$\begin{aligned}\left\langle \sigma(z_1)\,\partial X(z)\,\sigma(z_2)\,\partial\overline{X}(w)\,\sigma(z_3)\right\rangle &= \left\langle \sigma(z_1)\,\partial X_{\rm qu}(z)\,\sigma(z_2)\,\partial\overline{X}_{\rm qu}(w)\,\sigma(z_3)\right\rangle + \\ &+ \partial X_{\rm cl}(z)\,\partial\overline{X}_{\rm cl}(w)\,\langle \sigma(z_1)\,\sigma(z_2)\,\sigma(z_3)\rangle\,, \end{aligned} \tag{2.60}$$

and we deal with each portion separately.

Taking the quantum part first, define a function $g(z,w;z_i)$ as

$$g(z,w;z_i) = \frac{\left\langle \partial X_{\rm qu}(z)\,\partial\overline{X}_{\rm qu}(w)\prod_{i=1}^{3}\sigma(z_i)\right\rangle}{\left\langle \prod_{i=1}^{3}\sigma(z_i)\right\rangle}\,. \tag{2.61}$$

so that

$$\left\langle \sigma_1(z_1)\,\tau_2(z_2)\,\tau_3'(z_3)\right\rangle_{\rm qu} = \left\langle \prod_i \sigma(z_i)\right\rangle \lim_{\substack{z\to z_2\\ w\to z_3}} (z-z_2)^{\theta_2}\,(w-z_3)^{1-\theta_3}\, g(z,w;z_i)\,. \tag{2.62}$$

Since g is a function only of holomorphic variables, this result will be equally true on the torus or annulus; there is no need to take square roots anywhere except in the base twist correlator $\langle \prod_i \sigma_i (z_i) \rangle$.

An explicit expression for $g(z, w; z_i)$ is[3] [36]

$$g(z, w; z_i) = g_s(z, w) + \omega_1(w) \sum_{i=1}^{2} B_i \omega'_i(z) . \tag{2.63}$$

Using the result $\lim_{z\to 0} \vartheta_1(z) = \vartheta'_1(0) z$, we see

$$\begin{aligned} \lim_{w\to z_3} (w - z_3)^{1-\theta_3} \omega_i(w) &\propto \lim_{w\to z_3} (w - z_3)^{1-\theta_3} \vartheta_1(w - z_3)^{-(1-\theta_3)} \vartheta_1(w - z_3) \\ &= 0 , \end{aligned} \tag{2.64}$$

and so the second term in $g(z, w; z_i)$ disappears when we take the limit in equation (2.62). Therefore, the term of interest to us is $g_s(z, w)$, which is given by

$$g_s(z, w) = \gamma(w) \gamma'(z) \left[\frac{\vartheta'_1(0)}{\vartheta_1(w - z)}\right]^2 \sum_{i=1}^{3} \theta_i F_i(w, z) \vartheta_1(z - z_i) \prod_{j\neq i} \vartheta_1(w - z_j) , \tag{2.65}$$

with

$$F_i(w, z) = \frac{\vartheta_1\left(w - z + U_i^0\right)}{\vartheta_1\left(U_i^0\right)} \frac{\vartheta_1\left(w - z + Y_i - U_i^0\right)}{\vartheta_1\left(Y_i - U_i^0\right)} \tag{2.66}$$

$$Y_i = \left(\sum_{j=1}^{3} \theta_j z_j\right) - z_i , \tag{2.67}$$

and U_i^0 chosen so as to satisfy the equation $\partial_w F_i(w, z) = 0$. Taking the limit (2.62), one finds

$$\begin{aligned} \left\langle \sigma_1(z_1) \tau_2(z_2) \tau'_3(z_3) \right\rangle_{\text{qu}} = \left\langle \prod_i \sigma_i(z_i) \right\rangle \theta_3 F_3(z_3, z_2) \times \\ \times \vartheta_1(z_{12})^{-\theta_1} \vartheta_1(z_{13})^{\theta_1} \vartheta_1(z_{23})^{\theta_2-\theta_3-1} . \end{aligned} \tag{2.68}$$

To evaluate the classical part of (2.60) we insert (2.55) for the case of three twist operators,

$$\begin{aligned} \partial \overline{X}_{\text{cl}} &= \sum_{i=1}^{2} c_i \omega_i \\ \partial X_{\text{cl}} &= c_3 \omega'_3 , \end{aligned} \tag{2.69}$$

[3] As in the previous section, we deal with the $\theta \to 1 - \theta$ issue by exchanging $X \leftrightarrow \overline{X}$, which corresponds simply to swapping $z \leftrightarrow w$ relative to the work of Atick et al.

and take the limit (2.62) to find

$$
\begin{aligned}
\left\langle \sigma_1(z_1)\,\tau_2(z_2)\,\tau_3'(z_3)\right\rangle_{\rm cl} = \left\langle \prod_i \sigma_i(z_i)\right\rangle & \vartheta_1(z_{12})^{-\theta_1}\,\vartheta_1(z_{13})^{\theta_1-1}\,\vartheta_1(z_{23})^{\theta_2-\theta_3-1}\times\\
&\times\Big[c_1\vartheta_1(z_{23})\,\vartheta_1(-z_1(1-\theta_1)+z_2\theta_2+z_3\theta_3)+\\
&\quad + c_2\vartheta_1(z_{13})\,\vartheta_1(z_1\theta_1+z_2\theta_2-z_3(1-\theta_3))\Big]\times\\
&\times c_3\vartheta_1(z_1\theta_1-z_2(1-\theta_2)+z_3\theta_3)\,. \qquad (2.70)
\end{aligned}
$$

The factors of $\vartheta_1(z_{ij})^{\pm\theta}$ in both the quantum and classical parts of this excited twist correlator compensate exactly for the changes in the spin correlator introduced by picture-changing.

3. The quantum part

We are now in a position to construct the quantum part of A explicitly. In everything that follows it should be understood that the vertex operators z_i are positioned along the imaginary axis as shown, so that $z_i = \mathrm{i}y_i$. For ease of reference, we collect together the relevant correlators (2.30), (2.31), (2.43), (2.45), (2.57), (2.68) and (2.70) in figure 5.

Recall that in section 2, we found three contributions to A, which we denoted A_1, A_2 and A_3. We begin here with the term A_1, which we previously argued ought to correspond to a vertex renormalisation, limit 1 of figure 2. Using the correlators given, we find the rather involved expression

$$
\begin{aligned}
A_1 =\ & \frac{1}{4}\overline{u}_1 u_2 \mathrm{e}^{-S_{\rm cl}}\left(\sum_{i=1}^{3}\Phi^i(z_1,z_2,z_3)\right)\mathrm{e}^{\pi\left(k_1\cdot k_2 z_{12}^2+k_1\cdot k_3 z_{13}^2+k_2\cdot k_3 z_{23}^2\right)/t}\times\\
&\times\vartheta_1(z_{12})^{k_1\cdot k_2-\frac{3}{2}}\,\vartheta_1(z_{13})^{k_1\cdot k_3+\frac{1}{2}\left(\theta_1^1+\theta_1^2+\theta_1^3\right)-1}\,\vartheta_1(z_{23})^{k_2\cdot k_3+\frac{1}{2}\left(\theta_2^1+\theta_2^2+\theta_2^3\right)-3}\times\\
&\times\left(\prod_{i=1}^{3}\left|\det W^i\right|^{-\frac{1}{2}}\vartheta_1\left(\theta_1^i z_1+\theta_2^i z_2+(\theta_3^i-1)z_3\right)^{\frac{1}{2}}\vartheta_1(z_{12})^{\frac{1}{2}}\right)\times\\
&\times\sum_{\alpha\beta}\delta_{\alpha\beta}\vartheta_{\alpha\beta}\left(\frac{1}{2}(z_2-z_1)\right)\prod_{i=1}^{3}\vartheta_{\alpha\beta}\left(z_1 q_1^i+z_2\tilde{q}_2^i+z_3\tilde{q}_3^i\right). \qquad (3.1)
\end{aligned}
$$

The function $\Phi^i(z_1,z_2,z_3)$ in the first line comes from the excited twist correlators, and is displayed explicitly in figure 5. Factors in the second line come from the combination of all correlators, the third from the ordinary twist correlation and the fourth from the spin and ghost spin field correlators. The values of q_1^i, $\tilde{q}_2^i$, $\tilde{q}_3^i$ are given in table 1.

The phases $\delta_{\alpha\beta}$ may be determined by the requirement that as $z_1\to z_2\to z_3$, the amplitude ought to factor onto the partition function $\mathcal{Z}$ for two D6-branes, which contains the term

$$
\mathcal{Z}\propto\sum_{\alpha\beta}\delta_{\alpha\beta}\vartheta_{\alpha\beta}(0)^4 \qquad (\delta_{00}=\delta_{11}=+1,\ \delta_{01}=\delta_{10}=-1)\,. \qquad (3.2)
$$

$$\left\langle \prod_i \mathrm{e}^{\mathrm{i}k_i\cdot X}(z_i)\right\rangle = \prod_{i<j}\left[\vartheta_1(z_{ij})\,\mathrm{e}^{\pi z_{ij}^2/t}\right]^{k_i\cdot k_j}$$

$$\left\langle \partial X^\mu(w)\prod_i \mathrm{e}^{\mathrm{i}k_i\cdot X}(z_i)\right\rangle = \prod_{i<j}\mathrm{i}k_j^\mu\left[\frac{\theta_1'(w-z_j)}{\theta_1(w-z_j)}-\frac{2\pi}{t}|w-z_j|\right]\left[\vartheta_1(z_{ij})\,\mathrm{e}^{\pi z_{ij}^2/t}\right]^{k_i\cdot k_j}$$

$$\left\langle \prod_i \mathrm{e}^{\mathrm{i}a_iH}(z_i)\right\rangle = K_{\alpha\beta}\left[\prod_{i<j}\vartheta_1(z_i-z_j)^{a_ia_j}\right]\vartheta_{\alpha\beta}\left(\sum_i a_iz_i\right)$$

$$\left\langle \prod_i \mathrm{e}^{a_i\phi}(z_i)\right\rangle = K^g_{\alpha\beta}\left[\prod_{i<j}\vartheta_1(z_i-z_j)^{-a_ia_j}\right]\vartheta^{-1}_{\alpha\beta}\left(\sum_i a_iz_i\right)$$

$$\left\langle \prod_i \sigma_i(z_i)\right\rangle = \mathrm{e}^{-S_{\mathrm{cl}}}\left|\det W\right|^{-\frac{1}{2}}\vartheta_1(\theta_1z_1+\theta_2z_2-(1-\theta_3)z_3)^{\frac{1}{2}}\times$$

$$\times\vartheta_1(z_{12})^{\frac{1}{2}}\prod_{i<j}^{3}\vartheta_1(z_{ij})^{-\frac{1}{2}(1-\theta_i)(1-\theta_j)-\frac{1}{2}\theta_i\theta_j}$$

$$\langle\sigma(z_1)\,\tau(z_2)\,\tau'(z_3)\rangle = \left\langle \prod_i \sigma_i(z_i)\right\rangle\vartheta_1(z_{12})^{-\theta_1}\,\vartheta_1(z_{13})^{\theta_1}\,\vartheta_1(z_{23})^{\theta_2-\theta_3-1}\times$$

$$\times\Phi(z_1,z_2,z_3)$$

with

$$\Phi(z_1,z_2,z_3)\equiv\left\{\theta_3F_3(z_3,z_2)+\left[c_1\frac{\vartheta_1(z_{23})}{\vartheta_1(z_{13})}\vartheta_1(-z_1(1-\theta_1)+z_2\theta_2+z_3\theta_3)+\right.\right.$$

$$\left.\left.+c_2\vartheta_1(z_1\theta_1+z_2\theta_2-z_3(1-\theta_3))\right]c_3\vartheta_1(z_1\theta_1-z_2(1-\theta_2)+z_3\theta_3)\right\}$$

Figure 5: Correlators required for the calculation of $\mathcal{A}$, up to overall normalisation factors. The term $F_3(w,z)$ is given by (2.66).

The H-charge conservation rule $q_1^i+\tilde{q}_2^i+\tilde{q}_2^i=0$ guarantees that (3.1) does indeed have this property, and so the relative phases in A_1 must be the same as those in $\mathcal{Z}$. Therefore, we may apply the Riemann identity (A.5), with the result

$$A_1\propto\vartheta_1\left(\frac{1}{2}\sum_i z_i\left(1-\sum_j\theta_i^j\right)\right). \tag{3.3}$$

Using the $\mathcal{N}=1$ supersymmetry condition $\sum_j\theta_i^j=1$ and the result $\vartheta_1(0)=0$, we see that A_1 always vanishes in supersymmetric models. This appears to support our identification of A_1 as a vertex renormalisation, since those diagrams vanish in $\mathcal{N}=1$ theories by the non-renormalisation theorem.

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We now turn to the terms $A_2^{\pm}$. Note first that after correlators have been inserted explicitly, the main difference between (2.23) and (2.24) comes from the spin-dependent parts coming from the external space: A_2^+ contains a factor of

$$\vartheta_{\alpha\beta}^{-1}\left(\frac{1}{2}(z_2-z_1)\right)\vartheta_{\alpha\beta}\left(\frac{1}{2}(z_2-z_1)\right)\vartheta_{\alpha\beta}\left(z_3-\frac{1}{2}(z_2+z_1)\right),$$

whilst A_2^- contains a factor of

$$\vartheta_{\alpha\beta}^{-1}\left(\frac{1}{2}(z_2-z_1)\right)\vartheta_{\alpha\beta}\left(-\frac{1}{2}(z_2-z_1)\right)\vartheta_{\alpha\beta}\left(-z_3+\frac{1}{2}(z_2+z_1)\right).$$

Since the ϑ-functions are odd or even, the difference is superficial and the terms $A_2^{\pm}$ may be joined together to give one single term:

$$\begin{aligned}A_2 = {} & \frac{1}{2}\overline{u}_1 u_2 \mathrm{e}^{-S_{\mathrm{cl}}}\mathrm{e}^{\pi\left(k_1\cdot k_2 z_{12}^2+k_1\cdot k_3 z_{13}^2+k_2\cdot k_3 z_{23}^2\right)/t}\times\\ & \times\left[k_1\cdot k_3\left(\frac{\theta_1'(z_{12})}{\theta_1(z_{12})}+\frac{2\pi\mathrm{i}}{t}z_{12}\right)+k_3\cdot k_3\left(\frac{\theta_1'(z_{23})}{\theta_1(z_{23})}+\frac{2\pi\mathrm{i}}{t}z_{23}\right)\right]\times\\ & \times\vartheta_1(z_{12})^{k_1\cdot k_2-\frac{1}{2}}\vartheta_1(z_{13})^{k_1\cdot k_3+\frac{1}{2}\left(\theta_1^1+\theta_1^2+\theta_1^3\right)-2}\vartheta_1(z_{23})^{k_2\cdot k_3+\frac{1}{2}\left(\theta_2^1+\theta_2^2+\theta_2^3\right)-2}\\ & \times\left(\prod_{i=1}^{3}\left|\det W^i\right|^{-\frac{1}{2}}\vartheta_1\left(\theta_1^i z_1+\theta_2^i z_2+\left(\theta_3^i-1\right)z_3\right)^{\frac{1}{2}}\vartheta_1(z_{12})^{\frac{1}{2}}\right)\times\\ & \times\sum_{\alpha\beta}\delta_{\alpha\beta}\vartheta_{\alpha\beta}\left(-\frac{1}{2}z_1-\frac{1}{2}z_2+z_3\right)\prod_{i=1}^{3}\vartheta_{\alpha\beta}\left(\left(\frac{1}{2}-\theta_1^i\right)z_1+\left(\frac{1}{2}-\theta_2^i\right)z_2-\theta_3 z_3\right).\end{aligned} \tag{3.4}$$

In a supersymmetric set-up, A_2 may be seen to vanish in exactly the same way as A_1. However, as it has kinematic factors in front, it does not contribute significantly when the vertex operators are far apart and so we do not associate it with a vertex renormalisation.

More interesting are the terms $A_3^{\pm}$, which also join together to give

$$\begin{aligned}A_3 = {} & -\frac{1}{2}\overline{u}_1 u_2 \mathrm{e}^{-S_{\mathrm{cl}}}k_2\cdot k_3\mathrm{e}^{\pi\left(k_1\cdot k_2 z_{12}^2+k_1\cdot k_3 z_{13}^2+k_2\cdot k_3 z_{23}^2\right)/t}\times\\ & \times\vartheta_1(z_{12})^{k_1\cdot k_2-\frac{3}{2}}\vartheta_1(z_{13})^{k_1\cdot k_3+\frac{1}{2}\left(\theta_1^1+\theta_1^2+\theta_1^3\right)-1}\vartheta_1(z_{23})^{k_2\cdot k_3+\frac{1}{2}\left(\theta_2^1+\theta_2^2+\theta_2^3\right)-3}\times\\ & \times\left(\prod_{i=1}^{3}\left|\det W^i\right|^{-\frac{1}{2}}\vartheta_1\left(\theta_1^i z_1+\theta_2^i z_2+\left(\theta_3^i-1\right)z_3\right)^{\frac{1}{2}}\vartheta_1(z_{12})^{\frac{1}{2}}\right)\times\\ & \times\sum_{\alpha\beta}\delta_{\alpha\beta}\vartheta_{\alpha\beta}\left(-\frac{1}{2}z_1+\frac{3}{2}z_2-z_3\right)\prod_{i=1}^{3}\vartheta_{\alpha\beta}\left(\left(\frac{1}{2}-\theta_1^i\right)z_1+\left(\frac{1}{2}-\theta_2^i\right)z_2-\theta_3 z_3\right).\end{aligned} \tag{3.5}$$

Again the phases may be determined from factorisation on the partition function and so we may apply the Riemann identity, leading to

$$\begin{aligned}A_3 = {} & -\overline{u}_1 u_2 \mathrm{e}^{-S_{\mathrm{cl}}}k_2\cdot k_3\mathrm{e}^{\pi\left(k_1\cdot k_2 z_{12}^2+k_1\cdot k_3 z_{13}^2+k_2\cdot k_3 z_{23}^2\right)/t}\times\\ & \times\vartheta_1(z_{12})^{k_1\cdot k_2-\frac{3}{2}}\vartheta_1(z_{13})^{k_1\cdot k_3+\frac{1}{2}\left(\theta_1^1+\theta_1^2+\theta_1^3\right)-1}\vartheta_1(z_{23})^{k_2\cdot k_3+\frac{1}{2}\left(\theta_2^1+\theta_2^2+\theta_2^3\right)-3}\times\\ & \times\left(\prod_{i=1}^{3}\left|\det W^i\right|^{-\frac{1}{2}}\vartheta_1\left(\theta_1^i z_1+\theta_2^i z_2+\left(\theta_3^i-1\right)z_3\right)^{\frac{1}{2}}\vartheta_1(z_{12})^{\frac{1}{2}}\right)\times\end{aligned} \tag{3.6}$$

$$\times\vartheta_1\left(\frac{1}{2}\sum_i\left(1-\sum_j\theta_i^j\right)z_i+z_2-z_3\right)\prod_k\vartheta_1\left(\frac{1}{2}\sum_{i,j}M_{kj}\theta_i^jz_i+\frac{1}{2}(z_2-z_1-z_3)\right),$$

where the matrix M is

$$M_{kj}=\begin{pmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{pmatrix}.$$

This time, the result does *not* vanish in an $\mathcal{N}=1$ theory, which suggests that we should identify A_3 as a field renormalisation. Also, since A_3 has kinematic factors in front, it will be most significant when we take two vertex operators to be close together; therefore, we confirm the intuitive results of figure 2.

As a summary, the three terms that we have found have the following properties:

Term	Vanishes by Riemann identity?	Kinematic prefactors?
A_1	$\surd$	$\times$
A_2	$\surd$	$\surd$
A_3	$\times$	$\surd$

It appears that we have a problem if we want an $\mathcal{N}=1$ theory, since the only term which is not killed by a Riemann identity, A_3, appears to disappear on-shell via

$$k_2\cdot k_3=\frac{1}{2}m^2=0\,. \tag{3.7}$$

In fact, we shall see in section 6 that the integral over y_i yields poles which should be cancelled with the kinematic prefactor before going on-shell. In this manner finite answers are obtained. In non-supersymmetric theories, we will see that the structure of A_2 and A_3 (and in particular, the relative minus sign between them) is such that the poles in the combined expression cancel.

In the case where one of the branes go through an orbifold fixed point, the correlators of spin and twist operators will be modified [36]. However, the differences must only be in the spin-dependent terms, i.e. the last lines of (3.1), (3.4) and (3.5). In order that the amplitude factor onto the twisted partition function, these modifications must be exactly those that are made to the spin-dependent terms in the partition function, and so it is possible to write down the above expressions in the case of orbifold fixed points without repeating the calculation explicitly. The situation is similar in the presence of O-planes, where Möbius strip diagrams may be present.

4. The classical action

We now return to the question of how to treat the global monodromy conditions and extract the classical contribution to the action. The classical action for each pair of complex coordinate can be written

$$S_{\rm cl}=\frac{1}{4\pi\alpha'}\int \mathrm{d}^2z\,\left(\partial X_{\rm cl}\overline{\partial}\overline{X}_{\rm cl}+\overline{\partial}X_{\rm cl}\partial\overline{X}_{\rm cl}\right). \tag{4.1}$$

The linear decomposition of $X_{\rm cl}$ can be defined,

$$\Delta_\gamma X_{\rm cl} = v_a = W_a^i c_i \tag{4.2}$$

where the displacements v_a are determined from the global monodromy conditions (i.e. by comparing the displacement of X under combinations of twists that add up to zero), as discussed earlier. The coefficients c_i are to be determined from them;

$$c_i = v_a (W^{-1})_i^a \,. \tag{4.3}$$

From the definition of ΔX we see that we must have

$$\begin{aligned} \partial \overline{X}_{\rm cl} &= \sum_{i=1}^{2} c_i \omega_i \\ \partial X_{\rm cl} &= c_3 \omega_3' \,. \end{aligned} \tag{4.4}$$

Inserting into S_{cl} gives

$$S_{\rm cl} = \frac{1}{4\pi\alpha'} \left(\sum_{i,j}^{2} c_i c_j^* I_{ij} + |c_3|^2 I_{33} \right) \tag{4.5}$$

where

$$\begin{aligned} I_{ij} &= \int \mathrm{d}^2 z\, \omega^i \bar{\omega}^j \\ I_{33} &= \int \mathrm{d}^2 z\, |\omega^3| \,. \end{aligned} \tag{4.6}$$

In order to determine the I_{ij} we may perform a canonical dissection of the torus [36]. In this case the dissection is as shown in figure 6.

In terms of the cycles and two spurs shown in the figure, the I_{ij} are given by [36]

$$\begin{aligned} I_{ij} = & \oint_{\gamma_1} \mathrm{d}z\, \omega^i \oint_{\gamma_2} \mathrm{d}\bar{z}\, \bar{\omega}^j - \oint_{\gamma_2} \mathrm{d}z\, \omega^i \oint_{\gamma_1} \mathrm{d}\bar{z}\, \bar{\omega}^j + \\ & + \int_{C_1} \omega^i \int_{C_2} \bar{\omega}^j + \sum_{l=1}^{2} \frac{1}{(1 - e^{-2\pi i \theta_l})} \int_{C_l} \omega^i \int_{C_l} \bar{\omega}^j \,. \end{aligned} \tag{4.7}$$

Evaluating the contour integrals explicitly we find the relations

$$\begin{aligned} \int_{C_1} \bar{\omega}^3 &= \frac{-1}{1 - \mathrm{e}^{-2\pi \mathrm{i} \theta_2}} W_3^3 \\ \int_{C_2} \bar{\omega}^3 &= \frac{1}{1 - \mathrm{e}^{2\pi \mathrm{i} \theta_2}} W_3^3 \\ \int_{C_1} \omega^{i=1,2} &= \frac{-1}{1 - \mathrm{e}^{-2\pi \mathrm{i} \theta_2}} W_3^i \\ \int_{C_2} \omega^{i=1,2} &= \frac{1}{1 - \mathrm{e}^{2\pi \mathrm{i} \theta_2}} W_3^i \end{aligned} \tag{4.8}$$

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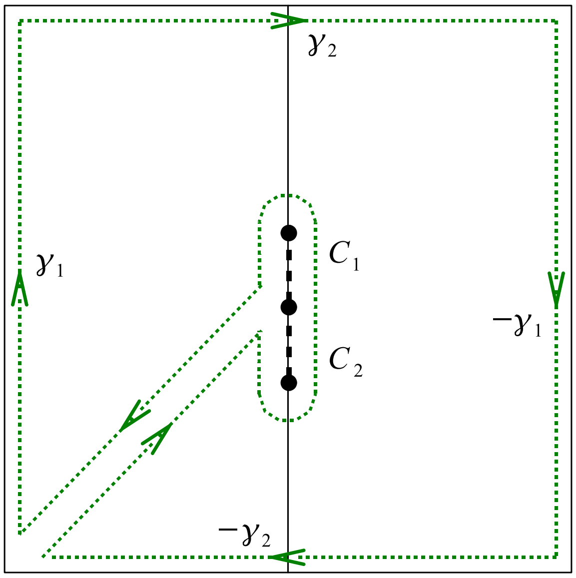

Figure 6: The dissection of the torus for three point diagrams.

whence we determine

$$I_{ij} = W_a^i \bar{W}_b^j K^{ab} \tag{4.9}$$

where

$$K^{ab} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & \alpha \end{pmatrix} \tag{4.10}$$

and

$$\alpha = \frac{1}{8\,\sin(\pi\theta_1)\sin(\pi\theta_2)\sin(\pi\theta_3)}\,. \tag{4.11}$$

This gives

$$4\pi\alpha' S_{\mathrm{cl}} = \sum_{i,j}^{2} c_i c_j^* W_a^i \overline{W}_b^j\, K^{ab} + c_3 c_3^* W_a^3 \overline{W}_b^3 K^{ab*} \tag{4.12}$$

where we have multiplied by an extra factor of $\frac{1}{2}$ to factor out half the world sheet after the Z_2 involution. Using the monodromy conditions this can be reduced to

$$4\pi\alpha' S_{\mathrm{cl}} = v_a v_b^* K^{ab} - c_3 W_a^3 v_b^* K^{ab} - c_3^* \overline{W}_b^3 v_a K^{ab} + 2|c_3|^2 W_a^3 \overline{W}_b^{3*} K^{ab*}\,. \tag{4.13}$$

Inserting K^{ab} yields

$$\begin{aligned} 8\pi\alpha' S_{\mathrm{cl}} &= \alpha\left(|v_3 - c_3 W_3^3|^2 + |c_3 W_3^3|^2\right) + \\ &\quad + \Big(\mathrm{i}\,|c_3|^2\, W_1^3 \overline{W}_2^{3*} - \mathrm{i}\,(v_1 - c_3 W_1^3)(v_2^* - c_3^* \overline{W}_2^3) \,+\, \mathrm{H.C.}\Big)\,. \end{aligned} \tag{4.14}$$

This is the main expression for the classical action. Note that we need only determine the coefficient c_3 explicitly; otherwise, all that remains to do is to find the W_a^j in the various limits and insert their values.

5. Limiting cases of the classical action

We are now in a position to apply the results derived thus far to the four limiting cases laid out in figure 2. We begin by looking at the classical action in these limits, which we expect to factor onto combinations of the classical contribution to the partition function,

$$\mathcal{Z}_{\rm cl} = {\rm e}^{-tY^2/2\pi\alpha'}\,, \tag{5.1}$$

and the tree-level Yukawa coupling,

$$\mathcal{Y}_{\rm cl} = {\rm e}^{-{\rm Area}/2\pi\alpha'}\,. \tag{5.2}$$

5.1 The partition function limit: $y_1 \to y_2 \to y_3$, $t \to \infty$

Consider first the factorisation of the classical part of the action, in a single T^2. In this limit, we expect to find

$$ {\rm e}^{-S_{\rm cl}} \to \mathcal{Z}_{\rm cl}\,\mathcal{Y}_{\rm cl}\,, \tag{5.3}$$

which is indeed the case. To see this, one must approximate the W integrals as $y_1 \to y_2 \to y_3$. Such an approximation is presented in appendix B.1, with the result

$$W = \begin{pmatrix} {\rm i}t & {\rm i}t & -{\rm i}t \\ 1 & 1 & 1 \\ W_3^1 & W_3^2 & W_3^3 \end{pmatrix}. \tag{5.4}$$

Inserting the first two rows into the monodromy conditions $v_a = W_a^i c_i$ gives

$$\begin{aligned} {\rm i}t\,(c_1 + c_2 - c_3) &= v_1 \\ c_1 + c_2 + c_3 &= v_2 \end{aligned} \tag{5.5}$$

with solution

$$c_3 = \frac{v_2}{2} - \frac{v_1}{2{\rm i}t}\,. \tag{5.6}$$

Since W_3^3 is vanishingly small in the limit (c.f. eq. (B.11)), we may also write

$$W_3^3 c_3 = 0\,. \tag{5.7}$$

Inserting these results into (4.14) leads to the action

$$2\pi\alpha' S_{\rm cl} = \frac{\alpha}{4}|v_3|^2 + t\left|\frac{v_2}{2}\right|^2 + \frac{|v_1|^2}{4t}\,. \tag{5.8}$$

Values must be inserted into this expression for the physical displacements v_i. First, note that v_3 is given by the Pochhammer contour

$$v_3 = 4\sin(\pi\theta_1)\sin(\pi\theta_2){\rm e}^{{\rm i}\pi(\theta_2-\theta_1)} f_{12}\,, \tag{5.9}$$

where f_{12} is the spacetime displacement between vertices one and two. Hence, the first term in the action is the area of a triangle

$$\frac{\alpha}{4}|v_3|^2 = \frac{\sin(\pi\theta_1)\sin(\pi\theta_2)|f_{12}|^2}{2\sin(\pi\theta_3)}\,, \tag{5.10}$$

and is the classical part of the tree level Yukawa coupling.

With our definitions, $|v_2|$ is twice the inter-brane separation in a given torus, $v_2 = 2\mathrm{i}Y_i$, since the contour integral goes across twice the fundamental domain of the annulus (c.f. figure 4). The second term in (5.8) is thus the partition function term $Y^2t/2\pi\alpha'$, seen in eq. (5.1).

From figure 1 we see that the displacement v_1 may be taken to zero. In a compact space, it should also be summed over all wrappings;

$$v_1 = 2\pi L n\,, \tag{5.11}$$

where $2\pi L$ is the wrapping length of the v_1 brane. Depending on the range of t under consideration, it may be appropriate to apply the Poisson resummation formula,

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi a n^2 + 2\pi \mathrm{i} b n} = a^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} \mathrm{e}^{-\pi(m-b)^2/a}\,, \tag{5.12}$$

after which we see that for a single torus

$$e^{-S_{\rm cl}} = \frac{\sqrt{2\alpha' t}}{L}\mathrm{e}^{-\mathrm{Area}/2\pi\alpha'}\mathrm{e}^{-tY^2/2\pi\alpha'} \sum_{m=-\infty}^{\infty} e^{-2\pi\alpha' t m^2/L}\,. \tag{5.13}$$

The extra factors of $\sqrt{t}$ here are due to Kaluza-Klein modes propagating in the loop. The condition for the resummation to be valid is that $2\pi\alpha' t > L$, and so we will get a factor of $\sqrt{t}$ appearing in the amplitude for each torus where $2\pi\alpha' t > L_i$. We discuss the consequences of this in section 6 below.

5.2 The vertex correction limit: $(t - y_3) \to \infty$, generic y_i

A derivation of the W integrals in this limit is presented in appendix B.2, where we show that up to terms which are suppressed by powers of t,

$$W = \begin{pmatrix} \mathrm{i}\Delta t + D & AC & -\mathrm{i}\Delta t + D + A \\ 1 & 1 & 1 \\ B & BC & B \end{pmatrix}, \tag{5.14}$$

with

$$\begin{aligned} A &= -\frac{\mathrm{i}}{2}\left[\cot(\pi\theta_1) + \cot(\pi\theta_3)\right] = -\frac{i}{2}\left(\frac{\sin(\pi\theta_2)}{\sin(\pi\theta_1)\sin(\pi\theta_3)}\right) \\ B &= -2\mathrm{i}e^{i\pi\theta_2}\sin(\pi\theta_2) \\ C &= -\frac{\Gamma(\theta_2)}{\Gamma(1-\theta_1)\Gamma(1-\theta_3)}e^{2\pi y_2\theta_1}\,, \end{aligned}$$

$$D = -\frac{\mathrm{i}}{2\pi}\left[2\gamma_E + \psi\left(1-\theta_1\right) + \psi\left(1-\theta_3\right)\right] \tag{5.15}$$

and $\Delta t = t - y_3$.

As the nine elements of W are expressible in terms of five independent quantities, we can see that there will be some over-determination in the monodromy conditions. That is, the equation

$$v_a = W_a^i c_i \tag{5.16}$$

will yield relations between the v_a. But recall that the v_a are physical displacements, relating the sides of triangles to their heights, and consequently the W_a^i matrix we have determined ought to give displacements corresponding to the actual Yukawa triangles. We will see that it does presently.

Notice first that since W_1^2 and W_3^2 contain the factor $e^{2\pi y_2 \theta_1}$, they are exponentially dominant in the v_1 and v_3 conditions and so we must have c_1, $c_3 \gg c_2$. The v_2 condition is therefore

$$c_1 + c_3 \simeq v_2\,. \tag{5.17}$$

Using this result together with the conditions for v_3 and v_1, we find

$$\begin{aligned} c_3 &= \frac{1}{2\mathrm{i}\Delta t - A}\left(\left(\mathrm{i}\Delta t + D - A\right)v_2 - v_1 + \frac{A}{B}v_3\right) \\ &\simeq \frac{v_2}{2} + \frac{\mathrm{i}}{2\Delta t}\left(v_1 - \frac{A}{B}v_3\right). \end{aligned} \tag{5.18}$$

To obtain the full S_{cl} via (4.1), we will need

$$c_3 W_3^3 = c_3 B = \frac{\hat{v}_2}{2} + \frac{\mathrm{i}A}{2\Delta t}\left(\hat{v}_1 - v_3\right), \tag{5.19}$$

where

$$\begin{aligned} \hat{v}_1 &= \frac{B}{A}v_1 \\ \hat{v}_2 &= B v_2\,. \end{aligned} \tag{5.20}$$

At this point it is appropriate to say something about the geometry of the situation. Notice that as A is pure imaginary, (5.19) shows that both $\hat{v}_1$ and $\hat{v}_2$ must have the same phase as v_3; arranging things so that v_1 is purely real and v_2 pure imaginary, v_3 must have an overall phase of $\mathrm{e}^{\mathrm{i}\pi\theta_2}$. However, since v_3 is also

$$v_3 = 4\sin(\pi\theta_1)\sin(\pi\theta_2)e^{i\pi(\theta_2-\theta_1)}f_{12}\,, \tag{5.21}$$

from the Pochhammer contour, f_{12} must have a phase of $\mathrm{e}^{\mathrm{i}\pi\theta_1}$. We may therefore define two additional vectors for $\hat{v}_1$ and $\hat{v}_2$, which we call f_1 and f_2 respectively, such that

$$\begin{aligned} \hat{v}_1 &= 4\sin\left(\pi\theta_1\right)\sin\left(\pi\theta_2\right)e^{\mathrm{i}\pi\left(\theta_2-\theta_1\right)}\,f_1 \\ \hat{v}_2 &= 4\sin\left(\pi\theta_1\right)\sin\left(\pi\theta_2\right)e^{\mathrm{i}\pi\left(\theta_2-\theta_1\right)}\,f_2\,. \end{aligned} \tag{5.22}$$

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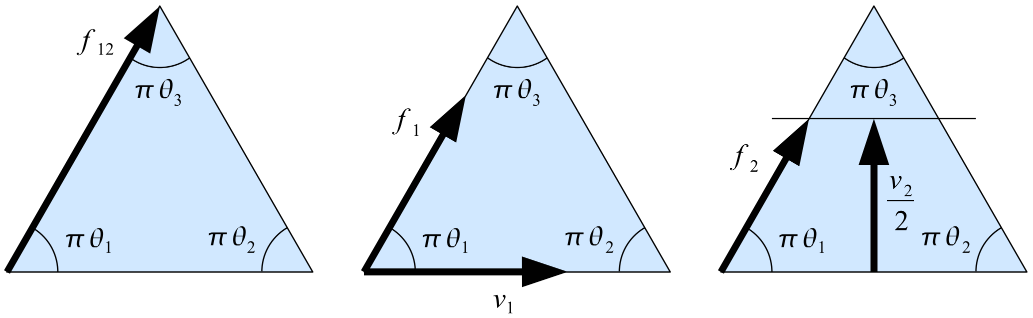

Figure 7: Physical vectors which appear in the global monodromy conditions $W_a^i c_i = v_a$.

With these definitions, f_1 and f_2 are projections of v_1 and $v_2/2$ along f_{12}:

$$\begin{aligned} f_1 &= \mathrm{e}^{\mathrm{i}\pi\theta_1} \frac{\sin(\pi\theta_3)\, v_1}{\sin(\pi\theta_2)} \\ f_2 &= \mathrm{e}^{\mathrm{i}\pi\left(\theta_1 - \frac{1}{2}\right)} \frac{v_2/2}{\sin(\pi\theta_1)}\,. \end{aligned} \tag{5.23}$$

The set-up is illustrated in figure 7.

Inserting (5.18), (5.19) into the result (4.1) and using $\alpha = \mathrm{i}A/|B|^2$ gives the classical action as

$$2\pi\alpha' S_{\mathrm{cl}} = \left|\frac{v_2}{2}\right|^2 (\Delta t + \mathrm{i}D) + \frac{\alpha}{4}\,|v_3 - \hat{v}_2|^2 + \frac{\alpha}{8}\,|\hat{v}_2|^2 + \mathcal{O}\left(\frac{1}{\Delta t}\right). \tag{5.24}$$

The first term in S_{cl} is the usual inter-brane suppression, $Y^2\Delta t/2\pi\alpha'$, showing that small $v_2 = 2\mathrm{i}Y$ is favoured. The second (and non-standard) term in S_{cl},

$$\frac{Y^2}{2\pi^2\alpha'}\left[2\gamma_E + \psi(1-\theta_1) + \psi(1-\theta_3)\right], \tag{5.25}$$

favours small values of θ_1 and θ_3, tending to zero as $\theta_1 \to 0$, $\theta_3 \to 0$; this is just another reflection of the desire of the string not to stretch.

As $v_2 \to 0$, the third term in S_{cl} become the area of the large triangle (c.f. equation (5.10)), and gives the tree level Yukawa coupling. The third term in the action is then the familiar effect of the propagation of a heavy (generation changing) mode of length $v_2/2$ stretched between two branes. This situation is shown in figure 8. The geometric interpretation of the $\frac{\alpha}{8}\,|\hat{v}_2|^2$ term is not obvious.

As $v_2 \to 0$, we expect to find Kaluza-Klein modes propagating in the loop. This may be seen by retaining the $\mathcal{O}(1/\Delta t)$ terms in S_{cl}, which when $v_2 = 0$ reads

$$2\pi\alpha' S_{cl} = \frac{\alpha}{4}|v_3|^2 + \frac{|v_1 - \tilde{v}_3|^2}{4\Delta t} + \mathcal{O}\left(\frac{1}{\Delta t^2}\right), \tag{5.26}$$

where

$$\tilde{v}_3 = \frac{A}{B} v_3 = e^{-i\pi\theta_1}\frac{\sin(\pi\theta_2)}{\sin(\pi\theta_3)} f_{12} \tag{5.27}$$

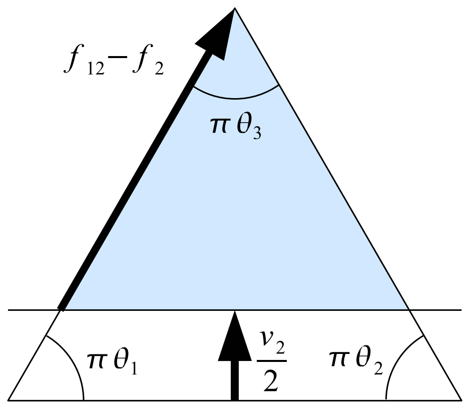

Figure 8: As $v_2 \to 0$, we recover the tree-level Yukawa coupling.

is the projection of v_3 along v_1. The displacement v_1 is allowed to wrap the brane between the 13 vertices, and we should sum over all wrappings. From figure 4, we see that v_1 must be at least as large as $\tilde{v}_3$ and so it is convenient to let

$$v_1 = 2\pi L n + \tilde{v}_3\,, \tag{5.28}$$

after which the resummation proceeds just as in section 5.1, with the result

$$e^{-S_{\rm cl}} = \frac{\sqrt{2\alpha'\Delta t}}{L} \mathrm{e}^{-\mathrm{Area}/2\pi\alpha'} \mathrm{e}^{-\Delta t Y^2/2\pi\alpha'} \sum_{m=-\infty}^{\infty} e^{-2\pi\alpha'\Delta t m^2/L}\,. \tag{5.29}$$

5.3 The purely twisted loop: $(t-y_3) \to 0$, generic y_i

As explained in appendix B.3, we now have

$$W = \begin{pmatrix} A + i\Delta t'_1 & AC & A - i\Delta t'_3 \\ 1 & 1 & 1 \\ B & BC & B \end{pmatrix} \tag{5.30}$$

with A, B and C defined as per the previous section, and

$$\begin{aligned} \Delta t'_1 &= \frac{\mathrm{i}}{2}\left[\frac{\pi^{\theta_1-1}}{\theta_1}(t-y_3)^{\theta_1} + \frac{\pi^{\theta_3-1}}{\theta_3}(t-y_3)^{\theta_3}\right] \\ \Delta t'_3 &= \frac{\mathrm{i}}{2}\left[\frac{\pi^{-\theta_1}}{1-\theta_1}(t-y_3)^{1-\theta_1} + \frac{\pi^{-\theta_3}}{1-\theta_3}(t-y_3)^{1-\theta_3}\right]. \end{aligned} \tag{5.31}$$

The same argument as in section 5.2 then leads to

$$c_3 = \frac{v_2}{1 + \Delta t'_3/\Delta t'_1} + \frac{\mathrm{i}}{\Delta t'_1 + \Delta t'_3}\left(v_1 - \frac{A}{B} v_3\right). \tag{5.32}$$

For $\theta_1 < \frac{1}{2}$ and $\theta_3 < \frac{1}{2}$, $\lim_{(t-y_3)\to 0} \Delta t'_3/\Delta t'_1 = 0$, and so the first term remains finite in the limit. The second term (and therefore $S_{\rm cl}$) tends to infinity in the limit, unless

$$v_1 = \frac{A}{B} v_3 = \tilde{v}_3\,, \tag{5.33}$$

in which case $c_3 = v_2$. Note that

$$\tilde{v}_3 = e^{-i\pi\theta_1} \frac{\sin \pi\theta_2}{\sin \pi\theta_3} f_{12} \tag{5.34}$$

is the projection of f_{12} along the v_1 direction, so this condition amounts to constraining our wrapping to triangles which are congruent to the original. As in the previous section, the origin of this constraint on v_1 lies in figure 4; in general, v_1 must be at least as large as $\tilde{v}_3$, but in the limit where we take vertex operators 1 and 3 to opposite sides of the worldsheet v_1 is pinched into a limit, leading to (5.33).

Substituting in to (4.14), one finds the classical action

$$2\pi\alpha' S_{\text{cl}} = 2 \left| \frac{v_2}{2} \right|^2 \Delta t_3' + \frac{\alpha}{4} \left| v_3 - \hat{v}_2 \right|^2 + \frac{3\alpha}{4} \left| \hat{v}_2 \right|^2 . \tag{5.35}$$

The first term, which goes to zero in the limit, is the stretching term already encountered. The second and third terms are area factors which give various combinations of Yukawa triangles. The three Yukawa couplings in the first diagram of limit 2 of figure 2 may be recovered by taking $v_3 = \hat{v}_2$. There are no terms suitable for Poisson resummation in this limit and so, correctly, Kaluza-Klein modes are not present.

5.4 The untwisted & twisted loop limit: $(t - y_3) \to \infty$, $y_2 \to y_1$

From appendix B.4, the W matrix in this limit is

$$W = \begin{pmatrix} \mathrm{i}\Delta t + E & \mathrm{i}\Delta t + E & -\mathrm{i}\Delta t - E \\ 1 & 1 & 1 \\ W_3^1 & W_3^2 & W_3^3 \end{pmatrix} \tag{5.36}$$

where

$$E = -\frac{\mathrm{i}}{2\pi} \left(2\gamma_E + \psi(\theta_3) + \psi(1 - \theta_3) \right) . \tag{5.37}$$

The monodromy conditions $v_a = W_a^i c_i$ now lead to

$$c_3 = \frac{v_2}{2} - \frac{v_1}{2 \left(\mathrm{i}\Delta t - E \right)} . \tag{5.38}$$

In this limit, $W_3^i \to 0$ (c.f. eq. (B.39)). Therefore, the action is

$$2\pi\alpha' S_{cl} = \left| \frac{v_2}{2} \right|^2 \left(\Delta t - \mathrm{i}E \right) + \frac{\alpha}{4} \left| v_3 \right|^2 + \frac{\left| v_1 \right|^2}{4 \left(\Delta t - \mathrm{i}E \right)} . \tag{5.39}$$

The first specimen is the standard string stretching suppression. The term in $-\mathrm{i}E$ has a minimum around $\theta_3 = \frac{1}{2}$ — like (5.25), it is also attempting to prevent the string stretching. The third term is the tree level Yukawa coupling; as we see from figure 2, no other couplings are present in this diagram. The fourth term may be Poisson resummed, after which the action takes the form

$$e^{-S_{\text{cl}}} = \frac{\sqrt{2\alpha' \left(\Delta t - \mathrm{i}E \right)}}{L} \mathrm{e}^{-\text{Area}/2\pi\alpha'} \mathrm{e}^{-(\Delta t - \mathrm{i}E) Y^2 / 2\pi\alpha'} \sum_{m=-\infty}^{\infty} e^{-2\pi\alpha' (\Delta t - \mathrm{i}E) m^2 / L} . \tag{5.40}$$

6. Extraction of β-functions

To complete our analysis, we will demonstrate how β-functions may be extracted by considering limit 3 — that is, the limit with both twisted and untwisted states in the loop. We concentrate on the field renormalisation terms A_2 and A_3.

6.1 Factorisation onto partition function

Recall that the amplitude takes the form

$$\mathcal{A} = \int_0^\infty \frac{\mathrm{d}t}{t} f(t) \int \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3\, A(z_1, z_2, z_3)\,. \tag{6.1}$$

The first thing that we must do is determine the overall normalisation $f(t)$ in (6.1). This may be done by the same mechanism used to determine phases previously in section 3: factorisation of the amplitude in the limit where the vertex operators come together (limit 0 of figure 2).

In the factorisation limit one expects to find $\mathcal{A} \to \mathcal{Z}\,\mathcal{Y}$. It was shown in section 5.1 that the classical part of the amplitude factorises correctly, so in what follows we concentrate on the quantum piece. For the case of D6-branes in a flat space, the quantum part of the partition function is, in the $t \to \infty$ limit,

$$\mathcal{Z}_{\text{qu}} = \int_0^\infty \frac{\mathrm{d}t}{4t} t^{-\frac{1}{2}(6+1)} \mathrm{e}^{\pi t} \sum_{\alpha\beta} \delta_{\alpha\beta} \vartheta_{\alpha\beta}(0)^4\ , \tag{6.2}$$

where we have ignored an infinite volume factor. We will also need the quantum normalisation of the tree-level Yukawa coupling, which is [49 – 52, 42, 48, 53]

$$\mathcal{Y}_{\text{qu}} = \prod_{i=1}^{3} \left[\frac{\Gamma(\theta_1^i)\,\Gamma(\theta_2^i)\,\Gamma(\theta_3^i)}{\Gamma(1-\theta_1^i)\,\Gamma(1-\theta_2^i)\,\Gamma(1-\theta_3^i)} \right]^{\frac{1}{4}}\ . \tag{6.3}$$

We now have nearly all the information required to write down the A_i terms in the $t \to \infty$ limit, save for one technicality: when we first began to construct correlators in terms of ϑ_1-functions in section 2.3, we mentioned that they ought to contain a contribution $r(t)$ which would regulate the correlators in the limit $z \to 0$. As we are now interested in the t-dependence of the correlators, we can no longer ignore this piece, which simply corresponds to making the replacement $\vartheta_1(z) \to \vartheta_1(z)/\vartheta_1'(0)$ everywhere, so that in the $t \to \infty$ limit the relevant expansion is just

$$\frac{\vartheta_1(\mathrm{i}y)}{\vartheta_1'(0)} \simeq \frac{\mathrm{i}\sinh(\pi y)}{\pi}\,. \tag{6.4}$$

No such replacement is made in the spin-structure dependent portion, as it does not originate from a physical propagator.

As explained in section 3, each of the terms A_1, A_2, A_3 factors correctly onto the partition function. We could therefore obtain $f(t)$ from any combination of these terms; we choose the field renormalisation limits A_2 and A_3, as it is a field renormalisation we will be ultimately interested in. To write down A_2 and A_3 in the limit, a result for $\det W$ as $y_3 \to y_2 \to 0$ is required. This is derived in appendix B.1 as

$$|\det W| = \frac{8\pi^2 t}{\Gamma(1-\theta_1)\Gamma(1-\theta_2)\Gamma(1-\theta_3)} y_{12}^{-\theta_3} y_{13}^{-\theta_2} y_{23}^{-\theta_1} y_2 \left(\theta_1 y_1 + \theta_2 y_2 - (1-\theta_3) y_3\right). \tag{6.5}$$

Notice that the factor of $y_2\left(\theta_2^i y_2 - \left(1-\theta_3^i\right) y_3\right)$ is exactly that required to make the twist correlator (2.57) symmetric in the y_i — which of course it should be, as it it does not distinguish between fermions and bosons.

The integrals are simplified if we use translational invariance on the worldsheet to fix $y_1 = 0$ (inserting a factor of t to compensate) and make the change of variables

$$\rho = \frac{y_2}{y_3} \qquad \lambda = \frac{t - y_3}{t}, \tag{6.6}$$

so that the integration over all vertex operator positions (2.6) is performed as

$$\mathcal{A}_2 + \mathcal{A}_3 = -\mathrm{i} \int_0^\infty \frac{\mathrm{d}t}{t} f(t)\, t^3 \int_0^1 \mathrm{d}\rho \int_0^1 \mathrm{d}\lambda\, (A_2 + A_3). \tag{6.7}$$

Inserting (6.4) and (6.5) into (3.4) and (3.5), one finds

$$\begin{aligned}\mathcal{A}_2 = &\int_0^\infty \frac{\mathrm{d}t}{t} f(t)\, g(t) \sum_{\alpha\beta} \delta_{\alpha\beta} \vartheta_{\alpha\beta}(0)^4 \times \\ &\times \Bigg[k_1{\cdot}k_3 \int_0^1 \mathrm{d}\rho\, \rho^{-1+k_1\cdot k_2+\frac{1}{2}m^2} (1-\rho)^{-1+k_2\cdot k_3-\frac{1}{2}m^2} \int_0^1 \mathrm{d}\lambda\, (1-\lambda)^{-1+k_1\cdot k_2+k_1\cdot k_3+k_2\cdot k_3-\frac{1}{2}m^2} + \\ &+ k_3^2 \int_0^1 \mathrm{d}\rho\, \rho^{k_1\cdot k_2+\frac{1}{2}m^2} (1-\rho)^{-2+k_2\cdot k_3-\frac{1}{2}m^2} \int_0^1 \mathrm{d}\lambda\, (1-\lambda)^{-2+k_1\cdot k_2+k_1\cdot k_3+k_2\cdot k_3-\frac{1}{2}m^2} \Bigg]\end{aligned} \tag{6.8}$$

and

$$\begin{aligned}\mathcal{A}_3 = &-\int_0^\infty \frac{\mathrm{d}t}{t} f(t) g(t) \sum_{\alpha\beta} \delta_{\alpha\beta} \vartheta_{\alpha\beta}(0)^4 \times \\ &\times k_2{\cdot}k_3 \int_0^1 \mathrm{d}\rho\, \rho^{-1+k_1\cdot k_2+\frac{1}{2}m^2} (1-\rho)^{-2+k_2\cdot k_3-\frac{1}{2}m^2} \int_0^1 \mathrm{d}\lambda\, (1-\lambda)^{-2+k_1\cdot k_2+k_1\cdot k_3+k_2\cdot k_3-\frac{1}{2}m^2}\end{aligned} \tag{6.9}$$

where

$$g(t) = t^{-\frac{3}{2}+k_1\cdot k_2+k_1\cdot k_3+k_2\cdot k_3-\frac{1}{2}m^2} \prod_{i=1}^{3} \left[\frac{\Gamma\left(1-\theta_1^i\right)\Gamma\left(1-\theta_2^i\right)\Gamma\left(1-\theta_3^i\right)}{8\pi^2} \right]^{\frac{1}{2}}, \tag{6.10}$$

and an irrelevant overall phase has been dropped. All explicit angular dependence in the exponents of (6.8), (6.9) has vanished via the relation $m^2 = \sum_i \theta_3^i - 1$.

On shell, the momenta obey $k_1^2 = 0$, $k_2^2 = 0$, $k_3^2 = -m^2$, and so

$$k_1 \cdot k_2 = -\frac{1}{2}m^2, \qquad k_1 \cdot k_3 = \frac{1}{2}m^2, \qquad k_2 \cdot k_3 = \frac{1}{2}m^2. \tag{6.11}$$

It is therefore clear that the integrals over ρ contain poles on-shell. We deal with these by writing the integrals as gamma functions which may then be expanded about the poles: in A_3, for instance,

$$\int_0^1 \mathrm{d}\rho\,\rho^{-1+k_1\cdot k_2+\frac{1}{2}m^2}(1-\rho)^{-2+k_2\cdot k_3-\frac{1}{2}m^2} = \frac{\Gamma\left(k_1\cdot k_2+\frac{1}{2}m^2\right)\Gamma\left(k_2\cdot k_3-\frac{1}{2}m^2-1\right)}{\Gamma\left(k_1\cdot k_2+k_2\cdot k_3-1\right)} \simeq \frac{\left(k_1\cdot k_2+k_2\cdot k_3\right)}{\left(k_1\cdot k_2+\frac{1}{2}m^2\right)\left(k_2\cdot k_3-\frac{1}{2}m^2\right)}\,. \tag{6.12}$$

Performing a similar expansion in A_2 and taking everything on-shell that does not lead to a pole, we end up with

$$\mathcal{A}_2+\mathcal{A}_3=\int_0^\infty \frac{\mathrm{d}t}{t} f(t)\,g(t)\sum_{\alpha\beta}\delta_{\alpha\beta}\vartheta_{\alpha\beta}(0)^4\,\frac{k_1\cdot k_2+k_2\cdot k_3}{k_1\cdot k_2+\frac{1}{2}m^2}\left[\frac{k_2\cdot k_3-k_1\cdot k_3}{k_2\cdot k_3-\frac{1}{2}m^2}-m^2\right]. \tag{6.13}$$

The $k_2\cdot k_3-k_1\cdot k_3$ pole occurs because of the relative minus sign between the A_2 and A_3 terms, which may be traced back to the results of the picture-changing in equations (2.23)–(2.26). The poles in the last two terms cancel when we go on-shell, yielding the finite result

$$\mathcal{A}_2+\mathcal{A}_3=\left(1-m^2\right)\int_0^\infty \frac{\mathrm{d}t}{t} f(t)\,g(t)\sum_{\alpha\beta}\delta_{\alpha\beta}\vartheta_{\alpha\beta}(0)^4\,. \tag{6.14}$$

Comparison of (6.14) with the partition function (6.2) and Yukawa normalisation (6.3) then shows that

$$f(t)=t^{-2}\mathrm{e}^{\pi t}\frac{1}{1-m^2}\mathcal{Y}_{\mathrm{qu}}\prod_{i=1}^{3}\left[\frac{\Gamma\left(1-\theta_1^i\right)\Gamma\left(1-\theta_2^i\right)\Gamma\left(1-\theta_3^i\right)}{8\pi^2}\right]^{-\frac{1}{2}}. \tag{6.15}$$

In a compact space, $\mathcal{Z}_{\mathrm{qu}}$ is modified by a power of $t^{\frac{3}{2}}$ as $t\to\infty$, since the Kaluza-Klein modes are quenched in this limit and ought to be resummed [47]. In this case one should also include a power of $t^{\frac{3}{2}}$ in (6.14) coming from the resummation (5.13), so there is no nett effect upon $f(t)$.

As a check, one may wish to verify that the same power of t is obtained when the term A_1 is factored onto the partition function. It is then necessary to consider the term $\Phi(z_1,z_2,z_3)$ which is present in A_1 (displayed explicitly in figure 5). Note that by (2.66),

$$\lim_{w\to z} F_3(w,z)=1 \tag{6.16}$$

so that the first term in Φ becomes θ_3. The second term in Φ vanishes by $\theta_1+\theta_2+\theta_3=1$, and hence one finds that

$$\lim_{z_1\to z_2\to z_3}\sum_{i=1}^{3}\Phi^i(z_1,z_2,z_3)=m^2+1\,. \tag{6.17}$$

Comparing (3.1) and (3.5), we see that the rest of A_1 is of the same form as A_3 in the factorisation limit; therefore, the correct power of t will be obtained.

6.2 The running coupling: logarithmic and power-law regimes

At last, we have manufactured all the ingredients necessary to evaluate Yukawa coupling renormalisation. We begin by considering the case of an $\mathcal{N}=1$ supersymmetric set-up, where the amplitude comes purely from the term A_3, and again work with pinching limit 3 of figure 2.

It is now necessary to include the effects of the spin-dependent theta functions $\vartheta_{\alpha\beta}$, which is most easily done by working with A_3 in the post-Riemann-identity form (3.6). Fixing $y_1=0$, making the replacement (6.4), using the result (B.40) for $\det W$ in the limit $y_2\to 0$ and taking the limit $y_2\to 0$ in the full expression (with $y_3\ll t$) leads to the following explicit form for A_3:

$$\begin{aligned}A_3 = & -\overline{u}_1 u_2 \mathrm{e}^{-S_{\mathrm{cl}}} k_2\cdot k_3 y_2^{k_1\cdot k_2-\frac{3}{2}} \sinh(y_3)^{k_1\cdot k_3+\frac{1}{2}(\theta_1^1+\theta_1^2+\theta_1^3)-1} \sinh(y_{23})^{k_2\cdot k_3+\frac{1}{2}(\theta_2^1+\theta_2^2+\theta_2^3)-3}\times\\ & \times\left[\prod_{i=1}^{3}\left(\frac{\Gamma\left(1-\theta_1\right)\Gamma\left(1-\theta_2\right)\Gamma\left(1-\theta_3\right)}{4\pi}\right)^{\frac{1}{2}}\times\right.\\ & \left.\quad\times\left(\left(2\pi y_2\right)^{1-\theta_3}\left(t-y_3\right)\right)^{-\frac{1}{2}}\sinh\left(\left(\theta_3^i-1\right)y_3\right)^{\frac{1}{2}} y_2^{\frac{1}{2}}\right]\times\\ & \times\mathrm{e}^{-\pi t}\sinh\left(\left(1+\theta_3^1+\theta_3^2+\theta_3^3\right)y_3\right)\sinh\left(\left(1+\theta_3^1-\theta_3^2-\theta_3^3\right)y_3\right)\times\\ & \times\sinh\left(\left(1-\theta_3^1+\theta_3^2-\theta_3^3\right)y_3\right)\sinh\left(\left(1-\theta_3^1-\theta_3^2+\theta_3^3\right)y_3\right). \end{aligned}\tag{6.18}$$

To cast (6.18) into a form that may be integrated, we note that it is dominated by large y_3, y_{23} and so we may approximate the sinh functions as exponentials, leading to the rather more pleasant result

$$A_3 \simeq \mathrm{e}^{-S_{\mathrm{cl}}} k_2\cdot k_3 h\left(t\right)\mathrm{e}^{\pi\left(k_1\cdot k_3+k_2\cdot k_3-m^2\right)y_3}\left(t-y_3\right)^{-\frac{3}{2}} y_2^{-1+k_1\cdot k_2+\frac{1}{2}m^2}, \tag{6.19}$$

where

$$h\left(t\right)=\mathrm{e}^{-\pi t}\left(2\pi\right)^{\frac{1}{2}-k_1\cdot k_3-k_2\cdot k_3+m^2}\prod_{i=1}^{3}\left[\frac{\Gamma\left(1-\theta_1^i\right)\Gamma\left(1-\theta_2^i\right)\Gamma\left(1-\theta_3^i\right)}{4\pi}\right]^{\frac{1}{2}} \tag{6.20}$$

with an overall phase dropped. The effect of $\mathrm{e}^{-S_{\mathrm{cl}}}$ needs to be taken into account; specifically, one must decide whether or not to perform the Poisson resummation (5.40). The condition for the resummation to be valid is that $2\pi\alpha'\Delta t\gg L$, so that the integral depends upon the value of $\Delta t=t-y_3$; ergo, the result obtained will depend upon the energy scale under consideration.

As $\Delta t\to\infty$, resummation of the Kaluza-Klein modes is appropriate. Then, the classical action (5.40) contributes a factor

$$\frac{\left(t-y_3\right)^{\frac{3}{2}}}{L_1 L_2 L_3} \tag{6.21}$$

to A_3 (we have set $\alpha'=\frac{1}{2}$ here). Choosing to make the change of variables (6.6), the amplitude integral reads

$$\mathcal{A}_3=-\mathrm{i}\int_{1/\Lambda^2}^{1/\mu^2}\frac{\mathrm{d}t}{t}f\left(t\right)t^3\int_0^1\mathrm{d}\rho\int_{1/t\Lambda'^2}^{1}\mathrm{d}\lambda\, A_3 \tag{6.22}$$

At this point we should say something about the cut-offs on the integrals. Firstly, note that we will require a UV cutoff Λ on our t-integral, despite the UV-finite nature of string theory. This cutoff is an artefact of making the large-t approximation, equivalent to sending $\alpha' \to 0$; if we were able to perform the appropriate elliptic integrals *without* making this approximation, we ought to find a UV-regular result. Secondly, we have placed a lower cut-off of $1/t\Lambda'^2$ on the λ integral. This may be viewed as a UV cutoff, since it removes the region where $y_3 \to t$. With a suitable choice for Λ', it also enforces the requirement that $2\pi\alpha'\Delta t \gg L$ for the Kaluza-Klein resummation to be valid.

With our change of variables, we have

$$\begin{aligned} \mathcal{A}_3 = & -\int_{1/\Lambda^2}^{1/\mu^2} \frac{\mathrm{d}t}{t} f(t)\, h(t)\, t^{2+k_1\cdot k_2+\frac{1}{2}m^2} \left(L_1L_2L_3\right)^{-1} k_2\cdot k_3 \times \\ & \times \int_0^1 \mathrm{d}\rho\, \rho^{-1+k_1\cdot k_2+\frac{1}{2}m^2} \int_{1/t\Lambda'^2}^1 \mathrm{d}\lambda\, \mathrm{e}^{\left(k_1\cdot k_3+k_2\cdot k_3-m^2\right)\pi t(1-\lambda)} \left(1-\lambda\right)^{k_1\cdot k_2+\frac{1}{2}m^2} . \end{aligned} \tag{6.23}$$

As in the previous section, the function of the ρ integral is to supply a pole corresponding to the internal propagator in figure 2. The λ integral contains no poles when taken on-shell, so we may impose the on-shell conditions (6.11) here *before* performing the integral. Sending the cutoff $\Lambda' \to \infty$,

$$\begin{aligned} \mathcal{A}_3 &= -\left(L_1L_2L_3\right)^{-1} \frac{k_2\cdot k_3}{k_1\cdot k_2+\frac{1}{2}m^2} \int_{1/\Lambda^2}^{1/\mu^2} \frac{\mathrm{d}t}{t} f(t)\, h(t)\, t^{2+k_1\cdot k_2+\frac{1}{2}m^2} \\ &= \mathcal{Y}_{\mathrm{qu}} \left(2\pi\right)^{-1} \left(L_1L_2L_3\right)^{-1} \int_{1/\Lambda^2}^{1/\mu^2} \frac{\mathrm{d}t}{t} \\ &\propto \log\left(\frac{\mu}{\Lambda}\right) . \end{aligned} \tag{6.24}$$

Where in the second line we have set $m^2 = 0$ and cancelled the poles before taking everything on-shell. As one would expect, the string calculation reproduces the usual logarithmic running of couplings at low energy.

At higher energy scales, t can become sufficiently small that resummation of the Kaluza-Klein modes is no longer appropriate: $2\pi\alpha'\Delta t \ll L_i$ for all L_i. The classical action then contributes no powers of Δt to the integral, which now reads

$$\begin{aligned} \mathcal{A}_3 = & -\int_{1/\Lambda^2}^{1/\mu^2} \frac{\mathrm{d}t}{t} f(t)\, h(t)\, t^{\frac{1}{2}+k_1\cdot k_2+\frac{1}{2}m^2} \times \\ & \times k_2\cdot k_3 \int_0^1 \mathrm{d}\rho\, \rho^{-1+k_1\cdot k_2+\frac{1}{2}m^2} \int_{1/t\Lambda'^2}^1 \mathrm{d}\lambda\, \mathrm{e}^{\left(k_1\cdot k_3+k_2\cdot k_3-m^2\right)\pi t(1-\lambda)} \left(1-\lambda\right)^{k_1\cdot k_2+\frac{1}{2}m^2} \lambda^{-\frac{3}{2}} . \end{aligned} \tag{6.25}$$

Notice that this differs from (6.23) by a power of $t^{-\frac{3}{2}}$. Also, the λ integral now has a pole as $\lambda \to 0$ (i.e. $y_3 \to t$), so that after performing the ρ integral, cancelling poles and taking everything on-shell,

$$\mathcal{A}_3 = \mathcal{Y}_{\mathrm{qu}} \left(2\pi\right)^{-1} \int_{1/\Lambda^2}^{1/\mu^2} \mathrm{d}t\, t^{-\frac{5}{2}} \int_{1/t\Lambda'^2}^1 \mathrm{d}\lambda\, \lambda^{-\frac{3}{2}} . \tag{6.26}$$

The IR and UV cutoffs in the two integrals should be associated with each other, so that

$$\mathcal{A}_3 \propto \mu^3 + \frac{3}{2}\Lambda^2\Lambda' . \tag{6.27}$$

We have obtained μ^3 running, as one would expect for D6-branes where three extra dimensions are present.

If the wrapping lengths L_i are of mixed sizes, the amplitude may be an admixture of the two regimes, with the power of μ in (6.27) dropping by one for each dimension which is not resummed. One may then envisage a situation where the power-law behaviour changes as the energy scale μ increases, beginning first with logarithmic running and then switching to power-law behaviour, with the power increasing as a greater number of extra dimensions open up (i.e. as $2\pi\alpha'\Delta t$ exceeds the brane wrapping lengths L_i in the various tori).

In a non-supersymmetric model, the above results are modified in two ways. Firstly, the terms A_1 and A_2 no longer vanish via the Riemann identity; therefore we have a vertex renormalisation term plus an additional source of field renormalisation. We will not evaluate the vertex renormalisation here (although we point out that sections 2-4 contain all of the necessary ingredients to do so), but instead examine how the field renormalisation is modified. The function $f(t)$ which was determined in section 6.1 is not modified in a non-supersymmetric model; the only difference is that we can no longer set m^2 there, so that the amplitudes A_2 and A_3 will now have a factor of $1 - m^2$ on the bottom.

Evaluation of the term A_3 proceeds exactly as above. The cancellation of poles in (6.24) and (6.26) appears now no longer valid, since we have $m^2 \neq 0$. However, including the term A_2 (which may be calculated in an analogous manner to A_3, save for an intermediate expansion in m^2) leads to an extra contribution which conspires to produce a cancellation of the same form as that in (6.13). This extra contribution cancels with the factor of $1-m^2$ in $f(t)$, so that the field renormalisation behaviour in non-supersymmetric models is the same as that in supersymmetric models.

In the case where one of the branes goes through an orbifold fixed point, or we have O-planes in our model, the comments at the end of section 3 apply — the spin-dependent portions of the A_i are modified. $f(t)$ should then be determined by factoring onto a *twisted* partition function, in which the modifications of the spin-dependent terms must be identical to those in the A_i. Up to an overall normalisation, then, $f(t)$ will be unchanged. The field renormalisation may then be evaluated by the procedure above, except that those portions of A_3 which come from spin-dependent terms (explicitly, the last two lines of (6.18)) must be appropriately modified.

7. Summary and discussion

In this work, we have performed the first one-loop string calculation of Yukawa couplings on intersecting branes. We began by developing the necessary technology for the calculation, in particular enumerating the selection rules on string states, describing the effects of picture-changing on the vertex operators and developing open-string correlators for the spin and twist fields. We then applied this technology to demonstrate the supersymmetric non-renormalisation theorem, and identified the vertex and field renormalisation contributions to the amplitude.

Following this, we derived an expression for the classical instanton part of the twist fields, before examining them in the various field-theory pinching limits of figure 2. In each limit, the classical action could be correctly factorised onto the appropriate tree-level Yukawa coupling(s) and partition function piece. The origin of Kaluza-Klein modes in those limits which contain bosons in the loop was identified. Finally we showed how Yukawa beta functions in intersecting brane models may be extracted from the calculation, finding that in the field-theory limit both logarithmic and power-law runnings are recovered depending upon the energy scale under consideration.

One point that we did not address is the quantum normalisation of the various limits. Note that the normalisation which enters the amplitude from the twist operators (via the $|\det W|^{-\frac{1}{2}}$ term) does not have the correct form (6.3), and that this must be inserted into the normalisation function $f(t)$ manually. Even so, one might expect that it ought to be possible to recover the correct normalisation factor by comparing the limits 2 and 3, which should contain factors of $\mathcal{Y}^3$ and $\mathcal{Y}$ respectively. Sadly, this is not the case: the problem is that limit 2 of figure 2 contains only twisted states in the loop, and thus ought to be factored onto a twisted partition function (a partition function between two branes at an angle). Strictly speaking, the $f(t)$ obtained is now *different*. It is then be necessary to include manually a $\mathcal{Y}^3$ factor, and our reasoning becomes circular. We conclude that our three-point calculation cannot be used to extract the quantum normalisation factor (6.3) — this is exactly the situation at tree-level, where the normalisation must be obtained by factorisation of a four-point correlator onto a three-point correlator [49 – 52, 38, 42, 48, 53].

As a final remark, we note that much of the conformal field theory framework developed in this paper is applicable to one-loop calculations on intersecting branes in general (and also in fact to orbifold models, providing an alternative to the calculations presented in [24]). In particular it would be interesting to apply our results to a four-point correlator at one-loop, which contains information about the one-loop Khler potential [53]. Furthermore, factorisation of this four-point amplitude onto a three-point amplitude should explicitly reproduce the Yukawa normalisation factor (6.3).

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A. The Jacobi ϑ-functions

We take the usual definition for the Jacobi ϑ-functions [54, 55],

$$\vartheta_{00}(z, \mathrm{i}t) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 t} \mathrm{e}^{2\mathrm{i}\pi n z}$$

$$= 1 + 2\mathrm{e}^{-\pi t} \cos(2\pi z) + 2\mathrm{e}^{-4\pi t} \cos(4\pi z) + \cdots \tag{A.1}$$

$$\vartheta_{01}(z, \mathrm{i}t) = \sum_{n=-\infty}^{\infty} (-1)^n \mathrm{e}^{-\pi n^2 t} \mathrm{e}^{2\mathrm{i}\pi n z}$$

$$= 1 - 2\mathrm{e}^{-\pi t} \cos(2\pi z) + 2\mathrm{e}^{-4\pi t} \cos(4\pi z) - \cdots \tag{A.2}$$

$$\vartheta_{10}(z, \mathrm{i}t) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi\left(n-\frac{1}{2}\right)^2 t} \mathrm{e}^{2\mathrm{i}\pi\left(n-\frac{1}{2}\right)z}$$

$$= 2\mathrm{e}^{-\pi t/4} \cos(\pi z) + 2\mathrm{e}^{-9\pi t/4} \cos(3\pi z) + \cdots \tag{A.3}$$

$$\vartheta_{11}(z, \mathrm{i}t) = \sum_{n=-\infty}^{\infty} (-1)^{n-\frac{1}{2}} \mathrm{e}^{-\pi\left(n-\frac{1}{2}\right)^2 t} \mathrm{e}^{2\mathrm{i}\pi\left(n-\frac{1}{2}\right)z}$$

$$= -2\mathrm{e}^{-\pi t/4} \sin(\pi z) + 2\mathrm{e}^{-9\pi t/4} \sin(3\pi z) + \cdots \tag{A.4}$$

We also define $\vartheta_1(z|\mathrm{i}t) = -\vartheta_{11}(z, \mathrm{i}t)$.

The ϑ-functions obey a useful ‘Riemann’ identity,

$$\sum_{\alpha\beta} \delta_{\alpha\beta} \vartheta_{\alpha\beta}(x) \vartheta_{\alpha\beta}(y) \vartheta_{\alpha\beta}(u) \vartheta_{\alpha\beta}(v) = 2\vartheta_1(x') \vartheta_1(y') \vartheta_1(u') \vartheta_1(v'), \tag{A.5}$$

where $\delta_{00} = \delta_{11} = +1$, $\delta_{01} = \delta_{10} = -1$ and

$$x' = \frac{1}{2}(x + y + u + v) \qquad u' = \frac{1}{2}(x + y - u - v)$$
$$y' = \frac{1}{2}(x - y + u - v) \qquad v' = \frac{1}{2}(x - y - u + v). \tag{A.6}$$

B. W integrals in various limits

Information about the geometry of the problem is contained in the W-integrals (2.58), which we examine in the $t \to \infty$ limit. Begin with integrals around γ_1, which we take to be along the imaginary axis $z = \mathrm{i}q$. Using the expansion (A.4) together with the result $\theta_1 + \theta_2 + \theta_3 = 1$, one can write the integrals up to exponentially suppressed terms as

$$W_1^1 = \mathrm{i}\int_{(y_3-t)/2}^{(y_3+t)/2} \mathrm{d}q\, (\mathrm{i}\sinh \pi q)^{\theta_1 - 1} (\mathrm{i}\sinh \pi(q - y_2))^{\theta_2} (\mathrm{i}\sinh \pi(q - y_3))^{\theta_3 - 1} \times$$
$$\times\, \mathrm{i}\sinh \pi(q + \theta_2 y_2 + (\theta_3 - 1) y_3) \tag{B.1}$$

$$W_1^2 = \mathrm{i}\int_{(y_3-t)/2}^{(y_3+t)/2} \mathrm{d}q\, (\mathrm{i}\sinh \pi q)^{\theta_1} (\mathrm{i}\sinh \pi(q - y_2))^{\theta_2 - 1} (\mathrm{i}\sinh \pi(q - y_3))^{\theta_3 - 1} \times$$
$$\times\, \mathrm{i}\sinh \pi(q + (\theta_2 - 1) y_2 + (\theta_3 - 1) y_3) \tag{B.2}$$

$$W_1^3 = -\mathrm{i}\int_{-(y_3+t)/2}^{-(y_3-t)/2} \mathrm{d}q\, \overline{(-\mathrm{i}\sinh \pi q)^{-\theta_1} (-\mathrm{i}\sinh \pi(q + y_2))^{-\theta_2} (-\mathrm{i}\sinh \pi(q + y_3))^{-\theta_3}} \times$$
$$\times\, \overline{-\mathrm{i}\sinh \pi(q + \theta_2 y_2 + \theta_3 y_3)}, \tag{B.3}$$

where the translational invariance of the problem has been used to set $y_1 = 0$ (in what follows we always assume $0 < y_1 < y_2 < t$). Our approximation has lost the periodicity of ϑ_1 in the imaginary direction; therefore, we have chosen limits on the integrals so as to keep the branch cuts in the centre of the worldsheet, ensuring that we integrate over one complete period.

Integrals around around γ_2 are simpler. Using the same approximation, one finds integrals of sine functions rather than sinh functions, taken along the real axis. In this case, we always find $W_2^i = 1$. Finally, integrals around γ_3 are of the same form as those around γ_1 but with the limits taken as 0 and y_2, and the overall result multiplied by a phase factor

$$P = 4\mathrm{e}^{\mathrm{i}\pi(\theta_2-\theta_1)} \sin(\pi\theta_2)\sin(\pi\theta_1)\,, \tag{B.4}$$

which comes from the Pochhammer loop [48].

B.1 Limit 0: $t \to \infty$, $y_1 \to y_2 \to y_3$

In this limit, we may expand (B.3) in y_1 and y_2 under the integrals, with the results

$$W_1^1 = W_1^2 = \mathrm{i}t + \mathcal{O}(y_i) \qquad W_1^3 = -\mathrm{i}t + \mathcal{O}(y_i)\,, \tag{B.5}$$

which give W the form

$$W \simeq \begin{pmatrix} \mathrm{i}t & \mathrm{i}t & -\mathrm{i}t \\ 1 & 1 & 1 \\ W_3^1 & W_3^2 & W_3^3 \end{pmatrix} \tag{B.6}$$

and hence

$$|\det W| = 2t\left(W_3^1 - W_3^2\right). \tag{B.7}$$

Trigonometric identities simplify this combination to

$$\begin{aligned} W_3^1 - W_3^2 &= \mathrm{i}P\sinh(\pi y_2)\sinh\left(\pi\left(y_2\theta_2 + y_3\left(\theta_3-1\right)\right)\right) \times \\ &\times \int_0^{y_2} \mathrm{d}q\, \left(\mathrm{i}\sinh\pi q\right)^{\theta_1-1}\left(\mathrm{i}\sinh\pi\left(q-y_2\right)\right)^{\theta_2-1}\left(\mathrm{i}\sinh\pi\left(q-y_3\right)\right)^{\theta_3-1}\,, \end{aligned} \tag{B.8}$$

and in the limit of small y_i the integral reduces to one which may be evaluated using the Gauss hypergeometric function ${}_2F_1$,

$$\begin{aligned} &\int_0^{y_2} \mathrm{d}q\, q^{\theta_1-1}\left(q-y_2\right)^{\theta_2-1}\left(q-y_3\right)^{\theta_3-1} = \\ &\qquad = \frac{\Gamma(\theta_1)\Gamma(\theta_2)}{\Gamma(\theta_1+\theta_2)}\,{}_2F_1\left(\theta_1, 1-\theta_3, \theta_1+\theta_2; \frac{y_2}{y_3}\right) y_2^{\theta_1+\theta_2-1} y_3^{\theta_3-1}\,. \end{aligned} \tag{B.9}$$

Including the phase factor P and using the condition $\theta_1+\theta_2+\theta_3 = 1$, plus the identity ${}_2F_1(a,b,b;z) = (1-z)^{-a}$, we end up with

$$|\det W| = \frac{8\pi^2 t}{\Gamma(1-\theta_1)\Gamma(1-\theta_2)\Gamma(1-\theta_3)} y_2^{-\theta_3} y_3^{-\theta_2}\left(y_3-y_2\right)^{-\theta_1} y_2\left(\theta_2 y_2 - \left(1-\theta_3\right)y_3\right). \tag{B.10}$$

If desired, we may reinstate y_1 by requiring that $\det W$ be translationally invariant; the result is shown in equation (6.5). We will also make use of W_3^3, which may be evaluated through use of hypergeometric functions as

$$W_3^3 = \frac{4\pi^2 y_{12}^{\theta_3} y_{13}^{\theta_2} y_{23}^{\theta_1}}{\Gamma(1-\theta_1)\Gamma(1-\theta_2)\Gamma(1-\theta_3)}\,. \tag{B.11}$$

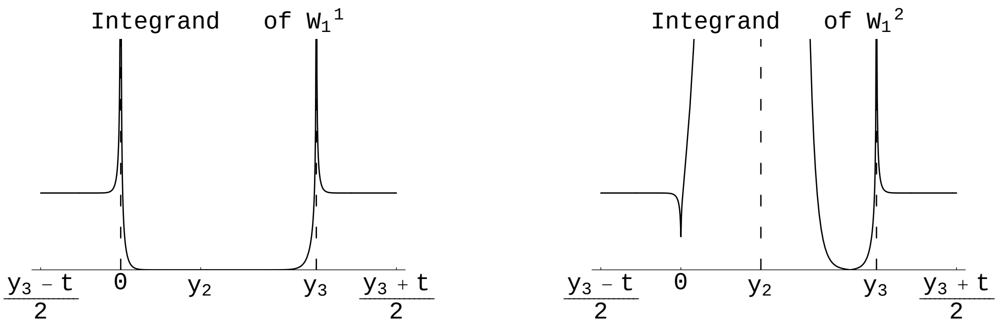

Figure 9: Absolute value of the W_1^i integrands, for arbitrarily chosen angles $0 < \theta_j < \frac{1}{2}$ and generic y_i. Not shown is W_1^3, which is simply a reflection of W_1^1 in the y-axis.

B.2 Limit 1: $(t - y_3) \to \infty$, generic y_i

The structure of our integrands in this case is shown in figure 9. The integrands of W_1^1 and W_3^1 both contain poles at $q = 0$ and $q = y_3$, but are regular elsewhere. Let us first consider integrals about γ_1. One may expand the W_1^1 integrand about the $q = 0$ pole as follows:

$$\begin{aligned} W_1^1\big|_0 &= \mathrm{i}\left(-\frac{1}{2}\right)^{1-\theta_1} \int_{(y_3-t)/2}^{\infty} \mathrm{d}q\ (\sinh \pi q)^{\theta_1 - 1}\, e^{\pi q(\theta_1 - 1)} \\ &= \mathrm{i}\frac{(-1)^{1-\theta_1}}{2\pi} B\left(\mathrm{e}^{\pi(t-y_3)}; 1-\theta_1, \theta_1\right) + 1\,, \end{aligned} \tag{B.12}$$

where $B\,(z; a, b)$ is the incomplete Euler Beta function. Expanding around the $q = y_3$ pole gives a similar result,

$$\begin{aligned} W_1^1\big|_{y_3} &= \mathrm{i}\left(\frac{1}{2}\right)^{1-\theta_3} \int_{-\infty}^{(y_3+t)/2} \mathrm{d}q\ (\sinh \pi\,(q - y_3))^{\theta_3 - 1}\, e^{\pi(q-y_3)(\theta_3 - 1)} \\ &= \mathrm{i}\left(\frac{1}{2}\right)^{1-\theta_3} \int_{-\infty}^{(t-y_3)/2} \mathrm{d}q'\ \left(\sinh \pi q'\right)^{\theta_3 - 1}\, e^{\pi q'(\theta_3 - 1)} \\ &= \mathrm{i}\frac{(-1)^{1-\theta_3}}{2\pi} B\left(\mathrm{e}^{\pi(t-y_3)}; 1-\theta_3, \theta_3\right). \end{aligned} \tag{B.13}$$

The full W_1^1 is then found as the sum of these two,

$$W_1^1 = \frac{\mathrm{i}}{2\pi}\left[(-1)^{1-\theta_1} B\left(\mathrm{e}^{\pi(t-y_3)}; 1-\theta_1, \theta_1\right) + (-1)^{1-\theta_3} B\left(\mathrm{e}^{\pi(t-y_3)}; 1-\theta_3, \theta_3\right)\right] + 1\,. \tag{B.14}$$

Since we are interested in the limit where $t - y_3 \to \infty$, we may apply the large-z expansion of the incomplete Euler Beta function,

$$B(z; a, b) = (-1)^{-a}\left(\mathrm{i}\pi + \log z - \psi\,(a) - \gamma_E\right) + \mathcal{O}\left(z^{-1}\right) \qquad (|z| \to \infty,\ a + b = 1)\,, \tag{B.15}$$

with the result

$$W_1^1 = \mathrm{i}\,(t - y_3) - \frac{\mathrm{i}}{2\pi}\left(2\gamma_E + \psi\,(1-\theta_1) + \psi\,(1-\theta_3)\right) + \mathcal{O}\left(t^{-1}\right). \tag{B.16}$$

Now consider the W_1^3 integral, which is similar. An analogous expansion about the poles at $q = 0$ and $q = -y_3$ leads to

$$W_1^3 = -\frac{\mathrm{i}}{2\pi}\left[(-1)^{\theta_1} B\left(\mathrm{e}^{\pi(t-y_3)};\theta_1, 1-\theta_1\right) + (-1)^{\theta_3} B\left(\mathrm{e}^{\pi(t-y_3)};\theta_3, 1-\theta_3\right)\right] - 1 \quad \text{(B.17)}$$

and using the large-z expansion (B.15),

$$W_1^3 = -\mathrm{i}\,(t-y_3) + \frac{\mathrm{i}}{2\pi}\left(2\gamma_E + \psi(\theta_1) + \psi(\theta_3)\right) + \mathcal{O}\left(t^{-1}\right). \quad \text{(B.18)}$$

The integrands for W_1^2 have poles at $q = y_2$ and $q = y_3$; however, the pole at y_2 is exponentially dominant for generic y_2. Therefore, the leading term in W_1^2 for generic y_2 is

$$\begin{aligned} W_1^2\Big|_{y_2} &= \mathrm{i}\,(-1)^{\theta_3}\, 2^{\theta_2-1}\mathrm{e}^{\pi y_2(1-\theta_2)}\int_{-\infty}^{\infty}\mathrm{d}q\,\left(\sinh\pi\,(q-y_2)\right)^{\theta_2-1}\mathrm{e}^{\pi q(\theta_1-\theta_3)} \\ &= \frac{\mathrm{i}\mathrm{e}^{2\pi y_2\theta_1}\Gamma(\theta_1)\,\Gamma(\theta_3)}{2\pi\Gamma(1-\theta_2)}. \end{aligned} \quad \text{(B.19)}$$

We could also extract the sub-leading behaviour, which we expect to be of a similar form to the result in W_1^1, but as we will see below it is unnecessary for our calculation.

Integrals around the Pochhammer cycle γ_3 may be found using the same technique of expanding about poles in q, but keeping only the portion of the integral which corresponds to the region $0 < q < y_2$. For instance, W_3^1 is found from the positive portion of the expansion around the first pole multiplied by the Pochhammer factor (B.4),

$$\begin{aligned} W_3^1 &= \mathrm{i}P\,W_1^1\Big|_{0,q>0} \\ &= -\mathrm{i}P\frac{(-1)^{\theta_1}}{2\,\sin(\pi\theta_1)} \\ &= -2\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\theta_2}\sin(\pi\theta_2). \end{aligned} \quad \text{(B.20)}$$

(The integrals W_1^i are along the principal branch, and so $(-1) = \mathrm{e}^{\mathrm{i}\pi}$.) Repeating the procedure for W_3^3 leads to an identical result,

$$W_3^3 = -2\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\theta_2}\sin(\pi\theta_2). \quad \text{(B.21)}$$

For generic y_2, the leading term in the W_3^2 integral is the portion of the $W_1^2\big|_{y_2}$ integral which is below the pole at $q = y_2$, multiplied by the Pochhammer factor,

$$\begin{aligned} W_3^2 &\simeq \mathrm{i}P\,W_1^2\Big|_{y_2,q<y_2} \\ &= \mathrm{i}P\frac{(-1)^{\theta_1}\,\mathrm{e}^{2\pi y_2\theta_1}\Gamma(\theta_1)\,\Gamma(\theta_2)}{2\pi\Gamma(1-\theta_3)} \\ &= \frac{2\mathrm{i}\pi\mathrm{e}^{\mathrm{i}\pi\theta_2}\mathrm{e}^{2\pi y_2\theta_1}}{\Gamma(1-\theta_1)\,\Gamma(1-\theta_2)\,\Gamma(1-\theta_3)}. \end{aligned} \quad \text{(B.22)}$$

Going from the second line to the third makes use of the reflection identity $\Gamma(\theta)\,\Gamma(1-\theta) = \pi\csc(\pi\theta)$.

It proves convenient to define a set of functions as follows,

$$
\begin{aligned}
A &= -\frac{\mathrm{i}}{2}\left[\cot\left(\pi\theta_1\right)+\cot\left(\pi\theta_3\right)\right] = -\frac{i}{2}\left(\frac{\sin(\pi\theta_2)}{\sin(\pi\theta_1)\sin(\pi\theta_3)}\right)\\
B &= -2\mathrm{i}e^{i\pi\theta_2}\sin(\pi\theta_2)\\
C &= -\frac{\Gamma(\theta_2)}{\Gamma(1-\theta_1)\Gamma(1-\theta_3)}e^{2\pi y_2\theta_1}\,,\\
D &= -\frac{\mathrm{i}}{2\pi}\left[2\gamma_E+\psi\left(1-\theta_1\right)+\psi\left(1-\theta_3\right)\right]
\end{aligned}
\tag{B.23}
$$

in terms of which the W matrix may be expressed as

$$
W = \begin{pmatrix} \mathrm{i}\Delta t + D & AC & -\mathrm{i}\Delta t + D + A \\ 1 & 1 & 1 \\ B & BC & B \end{pmatrix}, \tag{B.24}
$$

where $\Delta t = t - y_3$, we have dropped terms of order t^{-1} and also used the relation

$$
\psi\left(\theta\right) = \psi\left(1-\theta\right) - \pi\cot\left(\pi\theta\right).
$$

When we take the determinant, the dependence on $W_1^2 = AC$ drops out, leaving

$$
\begin{aligned}
|\det W| &= \left|B\left(1-C\right)\left(A+2D-2\mathrm{i}\Delta t\right)\right|\\
&\simeq 2\left|BC\right|\Delta t\,.
\end{aligned}
\tag{B.25}
$$

Explicitly, the leading term is

$$
|\det W| \simeq \frac{4\mathrm{e}^{2\pi y_2\theta_1}\left(t-y_3\right)}{\Gamma\left(1-\theta_1\right)\Gamma\left(1-\theta_2\right)\Gamma\left(1-\theta_3\right)}\,. \tag{B.26}
$$

The angular factor in front of this expression is identical to that found in (B.10).

B.3 Limit 2: $(t-y_3)\to 0$, generic y_i

As far as the W-integrals are concerned, the difference between this case and the first lies in only the integrals about γ_1. The expressions (B.14) and (B.17) for W_1^1 and W_1^3 are still valid in this limit, and their final form is obtained by using the expansion of the incomplete Euler beta function about $z=1$,

$$
B(z;a,b) = B\left(a,b\right) + \mathcal{O}\left(z^a\left(1-z\right)^b\right), \tag{B.27}
$$

with $B\left(a,b\right) = \Gamma\left(a\right)\Gamma\left(b\right)/\Gamma\left(a+b\right)$ the ordinary Euler beta function. We find

$$
\begin{aligned}
W_1^1 &= -\frac{\mathrm{i}}{2}\left[\cot\left(\pi\theta_1\right)+\cot\left(\pi\theta_3\right)\right] + \frac{\mathrm{i}}{2}\left[\frac{\pi^{\theta_1-1}}{\theta_1}\left(t-y_3\right)^{\theta_1} + \frac{\pi^{\theta_3-1}}{\theta_3}\left(t-y_3\right)^{\theta_3}\right] +\\
&\quad +\mathcal{O}\left(\left(t-y_3\right)^{1+\theta_i}\right),
\end{aligned}
\tag{B.28}
$$

and,

$$W_1^3 = -\frac{\mathrm{i}}{2}\left[\cot(\pi\theta_1) + \cot(\pi\theta_3)\right] - \frac{\mathrm{i}}{2}\left[\frac{\pi^{-\theta_1}}{1-\theta_1}(t-y_3)^{1-\theta_1} + \frac{\pi^{-\theta_3}}{1-\theta_3}(t-y_3)^{1-\theta_3}\right] + \\ +\mathcal{O}\left((t-y_3)^{2-\theta_i}\right). \tag{B.29}$$

The integral W_1^2 is unchanged from before, as are all integrals around γ_2 and γ_3. Writing

$$\begin{aligned} \Delta t_1' &= \frac{\mathrm{i}}{2}\left[\frac{\pi^{\theta_1-1}}{\theta_1}(t-y_3)^{\theta_1} + \frac{\pi^{\theta_3-1}}{\theta_3}(t-y_3)^{\theta_3}\right] \\ \Delta t_3' &= \frac{\mathrm{i}}{2}\left[\frac{\pi^{-\theta_1}}{1-\theta_1}(t-y_3)^{1-\theta_1} + \frac{\pi^{-\theta_3}}{1-\theta_3}(t-y_3)^{1-\theta_3}\right], \end{aligned} \tag{B.30}$$

and using the definitions (B.23), the W matrix is

$$W = \begin{pmatrix} A + i\Delta t_1' & AC & A - i\Delta t_3' \\ 1 & 1 & 1 \\ B & BC & B \end{pmatrix}. \tag{B.31}$$

Note that the t-independent terms in W_1^1 and W_1^3 now have the *same* sign, whereas in the previous section their signs were different. This means that they drop out of the determinant to leave

$$|\det W| \simeq \frac{4\mathrm{e}^{2\pi y_2\theta_1}\left(\Delta t_1' + \Delta t_3'\right)}{\Gamma(1-\theta_1)\,\Gamma(1-\theta_2)\,\Gamma(1-\theta_3)}. \tag{B.32}$$

B.4 Limit 3: $(t - y_3) \to \infty$, $y_2 \to y_1$

In this limit, an appropriate approach is to expand in y_2 under each integral and keep the leading term. For W_1^1, this gives

$$W_1^1 \simeq \mathrm{i}\int_{(y_3-t)/2}^{(y_3+t)/2} \mathrm{d}q\,\left(\mathrm{i}\sinh \pi q\right)^{-\theta_3}\left(\mathrm{i}\sinh \pi(q-y_3)\right)^{\theta_3-1}\mathrm{i}\sinh\pi\left(q + (\theta_3-1)\,y_3\right), \tag{B.33}$$

which has poles at $q = 0$ and $q = y_3$. As before, one may expand about these poles, integrate and use the expansion (B.15), leading to

$$W_1^1 = \mathrm{i}\,(t-y_3) - \frac{\mathrm{i}}{2\pi}\left(2\gamma_E + \psi(\theta_3) + \psi(1-\theta_3)\right) + \mathcal{O}\left(t^{-1}\right). \tag{B.34}$$

Other integrals about γ_1 are found in a similar fashion, with the result $W_1^1 = W_1^2 = -W_1^3$. Accordingly,

$$W = \begin{pmatrix} W_1^1 & W_1^1 & -W_1^1 \\ 1 & 1 & 1 \\ W_3^1 & W_3^2 & W_3^3 \end{pmatrix}, \tag{B.35}$$

and so

$$|\det W| = 2W_1^1\left(W_3^1 - W_3^2\right). \tag{B.36}$$

The integrals about γ_3 may performed by approximating them as, for instance,

$$\begin{aligned}W_3^1 &= \mathrm{i}P\left(-1\right)^{\theta_1-1}\left(2\pi\right)^{-\theta_3}\int_0^{y_2}\mathrm{d}q\,\mathrm{e}^{-\pi\theta_3(y_2-q)}q^{\theta_1-1}\left(y_2-q\right)^{\theta_2}\\ &= \mathrm{i}P\left(-1\right)^{\theta_1-1}\left(-2\pi\right)^{-\theta_3}y_2^{1-\theta_3}\mathrm{e}^{-\pi\theta_3 y_2}\int_0^1\mathrm{d}\lambda\,\mathrm{e}^{\pi\theta_3 y_2\lambda}\lambda^{\theta_1-1}\left(1-\lambda\right)^{\theta_2}\,,\end{aligned} \tag{B.37}$$

and using the integral

$$\int_0^1\mathrm{d}\lambda\,\mathrm{e}^{z\lambda}\lambda^{a-1}\left(1-\lambda\right)^{-a+b-1}=\frac{\Gamma\left(a\right)\Gamma\left(b-a\right)}{\Gamma\left(b\right)}+\mathcal{O}\left(z\right), \tag{B.38}$$

which may be found via the Kummer hypergeometric function ${}_1F_1$. The final results are

$$\begin{aligned}W_3^1 &= \frac{2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\theta_2}}{\Gamma\left(1-\theta_1\right)\Gamma\left(1-\theta_2\right)\Gamma\left(1-\theta_3\right)}\left(\frac{\theta_2}{1-\theta_3}\right)\left(2\pi y_2\right)^{1-\theta_3}\\ W_3^2 &= -\frac{2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\theta_2}}{\Gamma\left(1-\theta_1\right)\Gamma\left(1-\theta_2\right)\Gamma\left(1-\theta_3\right)}\left(\frac{\theta_1}{1-\theta_3}\right)\left(2\pi y_2\right)^{1-\theta_3}\\ W_3^3 &= \frac{2\pi\mathrm{i}\mathrm{e}^{\mathrm{i}\pi\theta_2}}{\Gamma\left(\theta_1\right)\Gamma\left(\theta_2\right)\Gamma\left(\theta_3\right)}\left(\frac{1}{\theta_3}\right)\left(2\pi y_2\right)^{\theta_3}\,,\end{aligned} \tag{B.39}$$

and the determinant therefore has the leading term

$$\left|\det W\right|\simeq\frac{4\pi\left(2\pi y_2\right)^{1-\theta_3}}{\Gamma\left(1-\theta_1\right)\Gamma\left(1-\theta_2\right)\Gamma\left(1-\theta_3\right)}\left(t-y_3\right). \tag{B.40}$$

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